N.M. Belyayev

Problems in Strength Of Materials

N. M. BELYAYEV

PROBLEMS IN STRENGTH OF MATERIALS

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ENGLISH TRANSLATION EDITED

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PUBLISHER'S NOTICE TO READERS

IN PUBLISHING an English translation of this book Pergamon Press has anticipated the eventual change-over to the metric system. However, until such a change-over does become effective the Publisher has rendered the book entirely suitable for readers used to working with British units by providing a comprehensive conversion table of the metric to British units and vice versa. This conversion table forms Section 1 of Appendix B.

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PREFACE TO THE SECOND EDITION

In compiling the second edition of this collection of problems the authors set themselves the task of keeping, as far as possible, to the trends adopted by the late N. M. Belyayev in the first edition. At the same time, however, they have ensured that the book is completely up to date, as far as the science of strength of materials is concerned.

In view of this, the first edition was substantially revised and considerably enlarged. Whilst the second edition contains many of the problems which were used in the previous edition, a large number of new problems has also been included. In addition to this, the authors thought it necessary to add several new sections reflecting the development of the science of strength of materials during recent years. In particular, the following sections have been introduced:

1. The calculation of thin-walled bars. The new methods were, in the main, developed by the Soviet scientists, V. Z. Vlasov, A. A. Umanskii and others, and increasing use is being made of them in engineering calculations.

2. The calculation of elements of structures and machines subjected to creep. These methods were extensively developed in the U.S.S.R. as a result of work carried out by N. M. Belyayev, A. A. Il'yushin, S. N. Serensen and others.

3. The determination of deformations and the calculation of statically indeterminate beams by the method of initial parameters. This method was also wholly developed by the Soviet scientists, N. P. Puzyrevskii, N. K. Snitko, A. N. Krylov and others.

Such sections in the book as the calculations on stability, oscillations, impact, fatigue and other phenomena, have undergone a certain amount of development.

In selecting the material the authors have tried to arrange the problems in order of increasing difficulty, being guided by their

maximum suitability as practical exercises for students working independently. In addition some problems have been introduced which are intended for use as set or independent homework. No especially complicated problems have been included in the collection.

Complete solutions have been provided for one or two problems of each type, whereas for most of the problems answers only have been given.

In solving problems we commend to the student the suggestion made by N. M. Belyayev in the preface to the first edition:

> In order to avoid a mechanical use of set formulae by students, the symbolical expressions for the initial data are omitted in many cases. In these cases the student himself must introduce these designations before commencing the solution, and then solve the problem in the algebraic form as long as this does not give rise to unnecessary complications. Only after this should he pass on to the numerical solution and complete it in that form. The result obtained should be checked by comparison with the initial data.

V. K. KACHURIN

PREFACE TO THE THIRD EDITION

IN THIS edition nearly every section of the collection has been enlarged with new problems, whilst certain duplications occurring in the preceding edition have been eliminated. The material contained in the collection has been rearranged and the presentation considerably simplified.

Answers have been supplied to a number of problems. A table of coefficients of allowable stresses has been added to the appendix. Defects discovered in the second edition have been eliminated.

The authors have received many comments from experts, regarding the preceding edition. In particular, a large number of valuable comments were received from L. P. Birichevskii, a senior instructor of the Strength of Materials Faculty of the Archangel Forestry Institute. The authors have taken these comments into full consideration, and express their sincere thanks to comrade L. P. Birichevskii.

The material was distributed amongst the individual authors in the following manner:

Sections 22–24 were compiled by L. A. Belyavskii; Section 35 by V. K. Kachurin; Sections 14–21 and 34 by Ya. I. Kipnis; Sections 10–13 and 29 by I. A. Kozhevnikov; Sections 1–7, 9, 33 and Appendix A by N. Yu. Kushelev; Sections 7, 8, 25–28 and 36–40 by A. K. Sinitskii.

V. K. KACHURIN

PREFACE
TO THE ENGLISH EDITION

THIS book of examples was written as a companion to the text book *Strength of Materials* by N. M. Belyayev (12th edition 1959). In view of the fact that some of the theoretical matter has been treated in a different manner from that usually encountered in English language texts, appropriate explanatory notes have been added in Sections 2, 3, and 4 of the specially written Appendix B, so that the relevant problems may be tackled.

Since many of the problems cannot be completed without reference to data (in the form of tables and diagrams) in the companion volume, the necessary information has been translated and included here in Appendix B, Section 6. In addition a comprehensive conversion table of metric units to British and American units has been included in Section 1 of the new appendix, in order that examples may be worked in the system of units desired. Finally, mechanical properties of certain materials mentioned in problems are given in Section 5 of Appendix B.

The wealth of carefully graded problems, with model answers for each topic, is the ideal arrangement for Higher National Diploma and Certificate students, and particularly those who are reading the subject with little or no supervision. While they are not all sufficiently advanced to cover the final year honours syllabuses in Strength of Materials for University students in Mechanical Engineering, the examples should amply illustrate the topics covered in most Universities and Colleges of Advanced Technology up to this level. The book should be particularly suitable for engineering students in departments where Strength of Materials is not a main subject, e.g. those studying electrical or chemical engineering. Some topics normally covered in Theory of Structures and Vibrations lectures are also included. The collection is perfectly suitable for use in conjunction with any standard work on Strength of Materials.

A number of minor mistakes were discovered during the translation of the book. These mistakes, in the main, were of the kind usually associated with type setting, and they have all been corrected in the translation.

R. KITCHING

DATA COMMON TO ALL PROBLEMS

UNLESS stated otherwise the following values of the quantities below must be taken while solving problems.

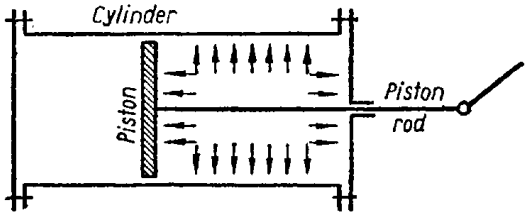
Modulus of elasticity of steel	$E = 2 \times 10^6$ kg/cm^2
Modulus of elasticity of aluminium and duralumin	$E = 0{\cdot}7 \times 10^6$ kg/cm^2
Modulus of elasticity of cast iron	$E = 1{\cdot}2 \times 10^6$ kg/cm^2
Modulus of elasticity of copper	$E = 1 \times 10^6$ kg/cm^2
Modulus of elasticity of timber (along the grain)	$E = 1 \times 10^5$ kg/cm^2
Modulus of rigidity of steel	$G = 8 \times 10^5$ kg/cm^2
Thermal linear expansion coefficient of steel	$\alpha = 125 \times 10^{-7}$/°C
Thermal linear expansion coefficient of copper	$\alpha = 165 \times 10^{-7}$/°C
Poisson's ratio of steel	$\mu = 0{\cdot}30$
Specific weight (weight/unit volume) of steel	$\gamma = 7{\cdot}8$ g/cm^3

CHAPTER 1

TENSION AND COMPRESSION

1. STATICALLY DETERMINATE SYSTEMS

1.1. The piston of a steam engine (see figure) has a diameter of 40 cm, whereas the diameter of the piston rod is 5·6 cm. The steam pressure is 10 atm (1 atm = 1 kg/cm²). Find the maximum stress in the piston rod and the corresponding change in its length during the stroke of the engine. The length of the rod is 75 cm; its material is steel.



PROBLEM 1.1

Solution. We shall refer to the cylinder diameter as $D(=40$ cm), the rod diameter as $d(=5{\cdot}6$ cm), the rod length as $l(=75$ cm), the steam pressure as $q(=10$ atm $=10$ kg/cm²) and the modulus of elasticity as $E(=2\times10^6$ kg/cm²).

The effective area of the piston

$$F_1=\frac{\pi}{4}(D^2-d^2)=0{\cdot}785(40^2-5{\cdot}6^2)=1230\text{ cm}^2.$$

The force in the rod

$$P=F_1q=1230\times10=12{,}300\text{ kg}.$$

The cross-sectional area of the rod

$$F_2=\frac{\pi d^2}{4}=\frac{\pi\times5{\cdot}6^2}{4}=24{\cdot}65\text{ cm}^2.$$

The stress in the rod

$$\sigma = \frac{P}{F_2} = \frac{12{,}300}{24{\cdot}65} = 500 \text{ kg/cm}^2,$$

and the elongation of the rod

$$\Delta l = \frac{Pl}{EF_2} = \frac{12{,}300 \times 75}{2 \times 10^6 \times 24{\cdot}65} = 0{\cdot}00187 \text{ cm} = 187 \times 10^{-4} \text{ cm}.$$

1.2. What is the maximum load a wooden pillar, with a cross-section of 16×16 cm^2, can support, without the compressive stress exceeding 100 kg/cm^2?

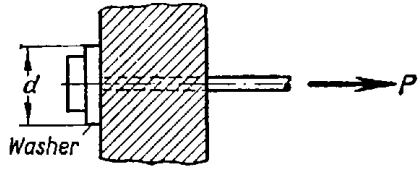
Answer: 25·6 t.

1.3. A cast iron column of annular cross-section has an external diameter of 25 cm and a wall thickness of 25 mm. What are the strain and contraction of the column under a load of 50 t? Determine the stress in the cross-section. The height of the column is 3 m.

Answer: $\varepsilon = 2{\cdot}36 \times 10^{-4}$, $\Delta l = 0{\cdot}71$ mm, $\sigma = -283$ kg/cm^2.

1.4. A tie bar 25 mm in diameter is pulled by a force P (see figure), inducing in it a stress of 1000 kg/cm^2. If the pressure transmitted to the wall by the washer is not to exceed 14 kg/cm^2, what must be the diameter d of the washer?

Answer: 213 mm.



PROBLEM 1.4

1.5. A copper wire 1·2 mm in diameter stretches 0·25 mm under a load of 9 kg. Determine the length of the wire.

Answer: 31·4 cm.

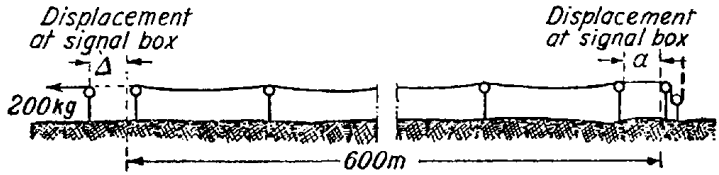
1.6. A cast iron column of annular cross-section has an external diameter of 30 cm, and is subjected to a force of 200 t. Determine

the necessary wall thickness, if the allowable compressive stress equals 800 kg/cm².

Answer: 2·9 cm.

1.7. A wire 5 mm in diameter and 600 m long, operating a railway signal, is laid on rollers as shown in the figure. Determine what displacement Δ, with a force of 200 kg, must be given to the end of the wire in the signal box, if the displacement a at the other end (attached to the signal) is to be 17·5 cm. Neglect the sag of the wire between the rollers, and the friction force between the wire and the rollers.

Answer: 48 cm.



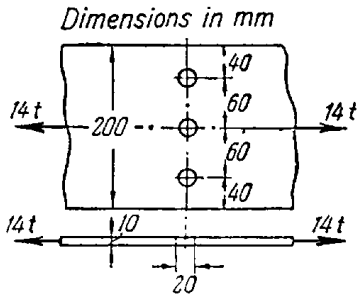
PROBLEM 1.7

1.8. A bar of low-carbon steel 30 cm wide and 15 mm thick is weakened by a rivet hole 23 mm in diameter, located on the axis of the bar. What tensile force can the bar withstand, if the allowable stress is 900 kg/cm²?

Answer: 37·4 t.

1.9. A steel plate (see figure) is subjected to longitudinal forces. It is weakened by rivet holes as shown in the figure. Determine the average value of the stresses in the critical section.

Answer: 1000 kg/cm².



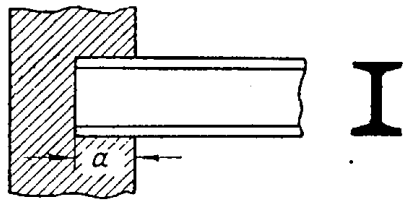
PROBLEM 1.9

1.10. The tie of an arched truss 10 m long sustains a pull of 60 t. The tie consists of two steel No. 18a channels (see tables in Appendix A). What is the elongation of the tie?

Answer: 5·8 mm.

1.11. The ends of a No. 30a I-beam are set in a brick wall (see figure). Through each end of the beam a load Q of 6 t is transmitted uniformly to the brickwork. Determine by what distance the beam must be set in the wall, if the allowable compressive stress $[\sigma]$ for the brickwork is 9 kg/cm^2. The width of the beam is to be taken from the tables in Appendix A.

Answer: 53 cm.



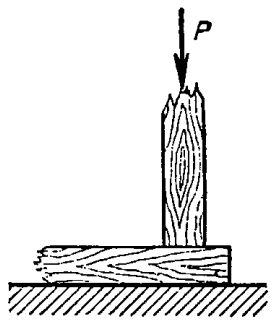
PROBLEM 1.11

1.12. Two wires, one of steel and the other of copper, have the same length and are subjected to equal axial tensile forces. The copper wire has a diameter of 1 mm. What is the diameter of the steel wire if both wires are elongated by the same amount?

Answer: 0·71 mm.

1.13. A pinewood stanchion with a 20 × 20 cm^2 cross-section rests on an oak pad as indicated in the figure. The allowable compressive stress $[\sigma_{\parallel}]$ along the grain for pine is 100 kg/cm^2; the allowable stress $[\sigma_{\perp}]$ across the grain for oak is 30 kg/cm^2. Determine the maximum load for the stanchion.

Answer: 12 t.



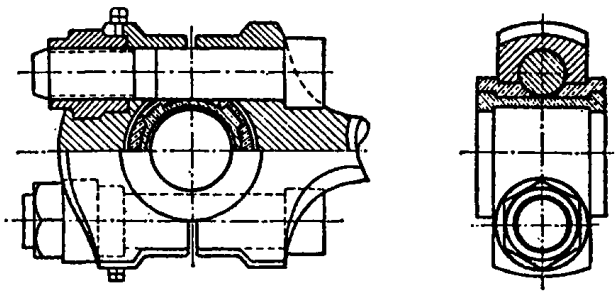
PROBLEM 1.13

1.14. A load P of 7·5 t is suspended from the lower end of a vertical cable. The cable is made up of wires 2 mm in diameter. The allowable stress $[\sigma]$ for the material of the cable is 3000 kg/cm^2. How many wires must there be in the cable?

Answer: 80 wires.

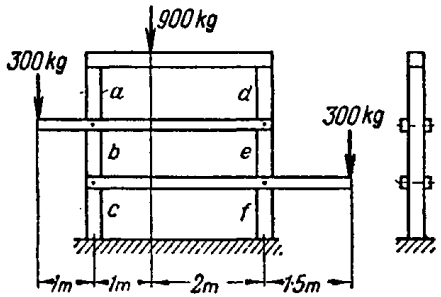
1.15. Determine the diameter of the two bolts joining the halves of the split end of a connecting rod (see figure); the force P in the connecting rod is 12,800 kg; the allowable stress $[\sigma]$ for the material of the bolts is 600 kg/cm^2.

Answer: 37 mm.



PROBLEM 1.15

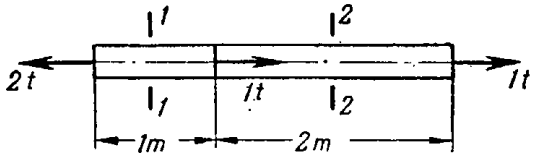
1.16. Two wooden posts, each with a cross-section of 10 × 10 cm^2, are loaded as shown in the figure. Determine the stress for both posts in the sections a, b, c, d, e, and f.



PROBLEM 1.16

Answer: Stresses, all in kg/cm^2: $\sigma_a = -6$; $\sigma_b = -10$; $\sigma_c = -8{\cdot}5$; $\sigma_d = -3$; $\sigma_e = -2$; $\sigma_f = -6{\cdot}5$.

1.17. Determine the stresses in the sections 1–1 and 2–2, and the total elongation of the steel rod shown in the figure. It is loaded as shown, and the area of its cross-section is 4 cm².



PROBLEM 1.17

Solution. Imagine the bar cut through the section 1–1 and the right-hand portion removed. To balance the force of 2 t applied to the remaining left-hand portion the resultant of the internal forces in the section 1–1 must also be 2 t and directed to the right. Thus, in the section 1–1 the force S_1 is tensile and equal to 2 t. By a similar reasoning it may be established that in the section 2–2 the force S_2 is also tensile, and has a magnitude of 1 t.

We can now determine the stresses. In section 1–1 the stress

$$\sigma_{1-1} = \frac{S_1}{F_1} = \frac{2000}{4} = 500 \text{ kg/cm}^2,$$

and in section 2–2

$$\sigma_{2-2} = \frac{S_2}{F_2} = \frac{1000}{4} = 250 \text{ kg/cm}^2.$$

Since the force in the left-hand portion (length 1 m) is not equal to that in the right-hand portion (length 2 m), the deformation of each portion must be determined separately. The total deformation of the rod is obtained by adding (algebraically) the deformations of the individual portions.

Thus we have

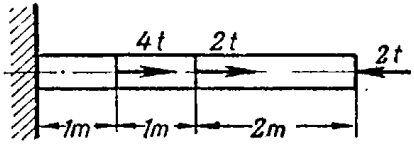
$$\Delta l = \Delta l_1 + \Delta l_2 = \frac{S_1 l_1}{EF} + \frac{S_2 l_2}{EF} =$$

$$= \frac{2000 \times 100}{2 \times 10^6 \times 4} + \frac{1000 \times 200}{2 \times 10^6 \times 4} = \frac{1}{40} + \frac{1}{40} =$$

$$= 0{\cdot}05 \text{ cm} = 0{\cdot}5 \text{ mm}.$$

1.18. Determine the stresses in all the three sections of the steel rod represented in the figure. Calculate also its total deformation if its cross-sectional area is 10 cm².

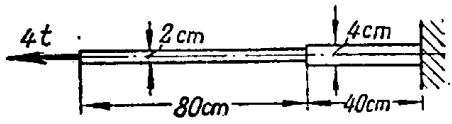
Answer: In the left-hand section, $\sigma = 400$ kg/cm²; in the middle section, $\sigma = 0$; in the right-hand section, $\sigma = -200$ kg/cm²; $\Delta l = 0$.



PROBLEM 1.18

1.19. Determine the stresses in both portions of the rod represented in the figure, and also its total elongation. The material of the rod, circular in cross-section, is steel.

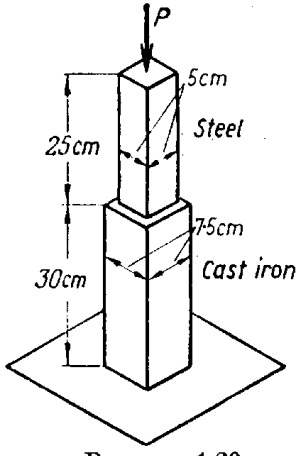
Answer: In the left-hand portion, $\sigma = 1276$ kg/cm²; in the right-hand portion, $\sigma = 319$ kg/cm²; $\Delta l = 0{\cdot}575$ mm.



PROBLEM 1.19

1.20. The upper part of the arrangement shown in the figure is of steel, whereas its lower part is of cast iron. The axial load P shortens the overall length by 0·2 mm. Determine the magnitude of the load P.

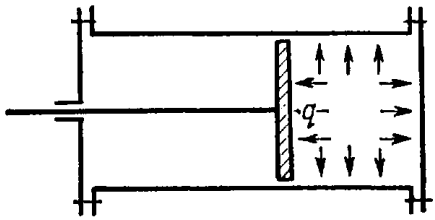
Answer: 21·2 t.



PROBLEM 1.20

1.21. The working pressure q in the cylinder of a steam engine is 10 atm (gauge); the internal diameter D of the cylinder is 350 mm (see figure). How many 18 mm diameter bolts are necessary to fasten the cover to the body of the cylinder, if the allowable stress $[\sigma]$ for the material of the bolts is 400 kg/cm²?

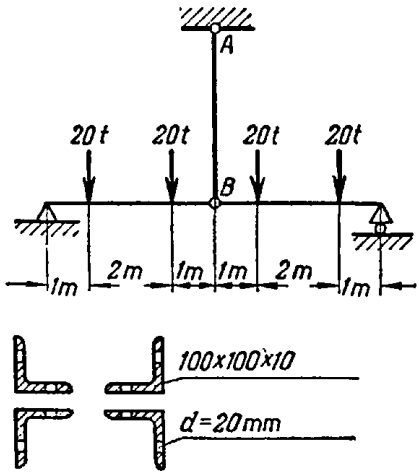
Answer: 10 bolts.



PROBLEM 1.21

1.22. The section of the hanger AB (see figure) consists of four standard 100 mm × 100 mm × 10 mm angles weakened by eight rivet holes, each 20 mm in diameter. Determine the stress in the critical cross-section of the hanger.

Answer: 659 kg/cm².

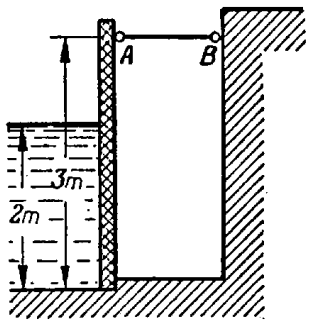


PROBLEM 1.22

1.23. A water-tight shield is prevented from overturning under water pressure by means of wooden struts AB (see figure). These

struts are provided every 3 m. Determine a suitable circular cross-section for the strut, if the allowable compressive stress $[\sigma]$ for the timber being used is 30 kg/cm^2.

Solution. Consider a section of the shield, having a length (perpendicular to the plane of the figure) equal to 3 m. There is one strut to this section. The thrust of water acting on the section considered is $3 \times (2 \times 2)/2 = 6$ t. Its direction is horizontal, and it is applied at a height equal to one third of the water depth, i.e. at two thirds of a metre from the bottom.



PROBLEM 1.23

The thrust of the shield on the strut at point A, equal to the compressive force in the strut, will be found by taking moments:

$$P = 6 \times \frac{2}{3 \times 3} = \frac{4}{3} \text{ t}.$$

The required area of the cross-section of the strut

$$F = \frac{P}{[\sigma]} = \frac{4000}{3 \times 30} = 44{\cdot}5 \text{ cm}^2,$$

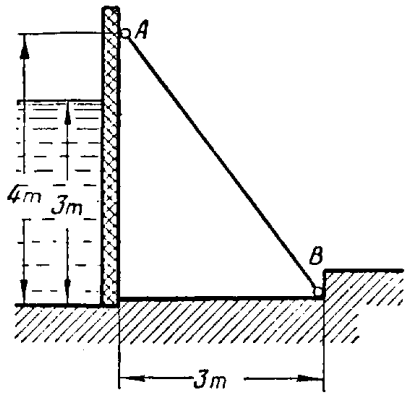
whence the diameter of the strut

$$d \geq 2\sqrt{\frac{F}{\pi}} = 2 \times \sqrt{\frac{44{\cdot}5}{3{\cdot}14}} = 7{\cdot}5 \text{ cm}.$$

1.24. A water-tight shield is prevented from overturning by means of wooden struts AB (see figure) each having a circular cross-section

15 cm in diameter. The allowable stress for the material of the struts is 20 kg/cm². Determine the maximum allowable distance between the struts.

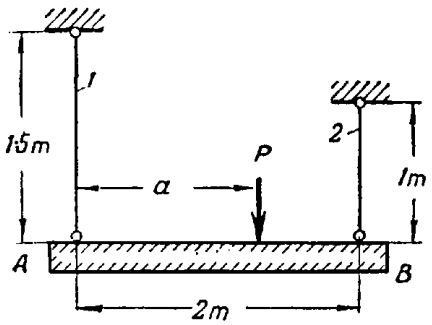
Answer: 1·88 m.



PROBLEM 1.24

1.25. A rigid beam AB, the deformation of which may be neglected, is suspended horizontally from rods 1 and 2, each of which has a circular cross-section. Rod 1 is of steel, and has a diameter of 20 mm, while rod 2 is of copper, and has a diameter of 25 mm. At what distance from the joint A (see figure) must the load P be applied, if the beam AB is to remain horizonatal after the deformation of the rods? What stresses will then occur in the rods, if $P = 3$ t?

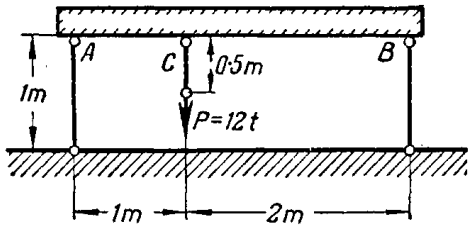
Answer: $a = 1{\cdot}08$ m; $\sigma_{(1)} = 440$ kg/cm²; $\sigma_{(2)} = 330$ kg/cm².



PROBLEM 1.25

1.26. A rigid beam AB (the deformation of which may be neglected) is supported on stanchions, and loaded as indicated in the figure. The stanchion A is of steel, with a cross-sectional area of 10 cm²; the stanchion B is of wood, with a cross-sectional area of 100 cm²; the rod C is of copper, and has a cross-sectional area of 30 cm². Determine the displacement of the point of application of the load.

Answer: $\Delta = 0{\cdot}6$ mm.



PROBLEM 1.26

1.27. Three rods securing a rigid structure to the foundation (see figure (a)) are made of steel and have identical cross-sectional areas of 20 cm² each. Determine the stresses in the rods, and also the horizontal, the vertical, and the total displacement of the point B, due to the action of the horizontal force.

Solution. For the determination of the forces in the rods we imagine these cut, and the forces, yet unknown, applied at the cutting points—assuming them all to be tensile (see figure (b)). We then set up the equations of equilibrium for these forces and the load P:

$$\sum X = P - S_{DB} = 0,$$

whence

$$S_{DB} = P;$$

$$\sum m_B = (P \times 2) + (S_{CF} \times 1) = 0,$$

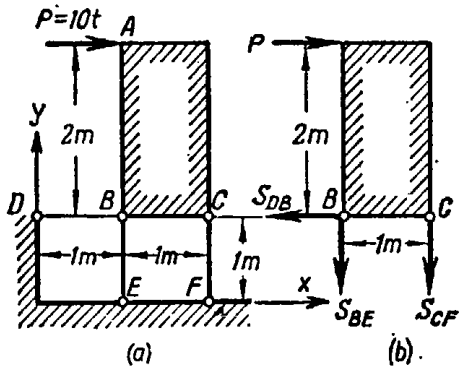
whence

$$S_{CF} = -2P;$$

$$\sum m_C = (P \times 2) - (S_{BE} \times 1) = 0,$$

whence

$$S_{EB} = 2P.$$



PROBLEM 1.27

Thus, for the rods DB and BE, our assumption made about the sign of the forces has proved to be correct, whereas for the rod CF it has not.

Knowing the magnitude of the forces it is possible to calculate the stresses in the rods:

$$\sigma_{DB} = \frac{S_{DB}}{F} = \frac{P}{F} = \frac{10{,}000}{20} = 500 \text{ kg/cm}^2,$$

$$\sigma_{BE} = \frac{S_{BE}}{F} = \frac{2P}{F} = \frac{2 \times 10{,}000}{20} = 1000 \text{ kg/cm}^2,$$

$$\sigma_{CF} = \frac{S_{CF}}{F} = -\frac{2P}{F} = -\frac{2 \times 10{,}000}{20} = -1000 \text{ kg/cm}^2.$$

The horizontal displacement of the point B equals the elongation of the rod BD. Consequently,

$$\Delta l_{hor.} = \Delta l_{BD} = \frac{S_{BD} l}{EF} = \frac{10{,}000 \times 100}{2 \times 10^6 \times 20} = 0{\cdot}025 \text{ cm} = 0{\cdot}25 \text{ mm}.$$

The vertical displacement of the point B equals the elongation of the rod BE. Consequently,

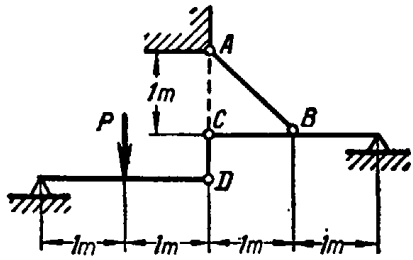
$$\Delta l_{\text{vert.}} = \Delta l_{\text{BE}} = \frac{S_{\text{BE}} l}{EF} = \frac{20{,}000 \times 100}{2 \times 10^6 \times 20} = 0{\cdot}05 \text{ cm} = 0{\cdot}5 \text{ mm}.$$

The total displacement of the point B will be obtained by the geometrical addition of the horizontal and the vertical displacements:

$$\Delta = \sqrt{\Delta^2_{\text{hor.}} + \Delta^2_{\text{vert.}}} = \sqrt{0{\cdot}25^2 + 0{\cdot}5^2} = 0{\cdot}56 \text{ mm}.$$

1.28. Select the diameters of the ties AB and CD (see figure) if the load P is 10 t, and the allowable stress $[\sigma]$ for the material of the ties is 1000 kg/cm^2.

Answer: $d_{\text{AB}} = 42{\cdot}4$ mm; $d_{\text{CD}} = 25.2$ mm.

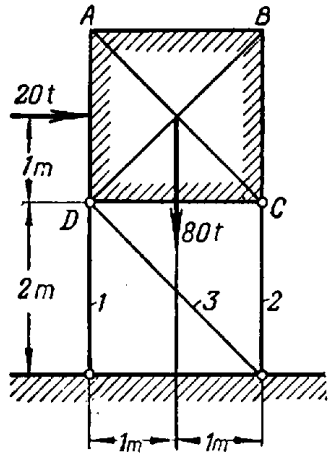


PROBLEM 1.28

1.29. A rigid structure ABCD is supported by two uprights 1 and 2 and a diagonal member 3, pin-jointed together (see figure). The structure weighs 80 t and is subject to a side thrust of 20 t. Select suitable cross-sections for the uprights and the diagonal member, each to be made up of four standard equal angles, if $[\sigma]$ is 1000 kg/cm^2. (See tables in Appendix A.)

Answer: The cross-section of the upright 1 should be made up of 35 mm × 35 mm × 4 mm angles; the cross-section of the upright 2, of 80 mm × 80 mm × 8 mm angles; and the cross-

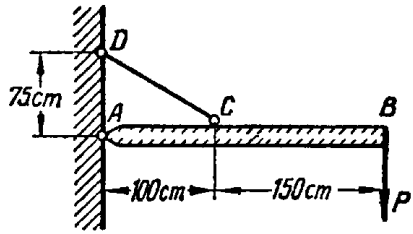
section of the diagonal member 3, of 65 mm × 65 mm × 6 mm angles.



PROBLEM 1.29

1.30. A rigid bar AB (see figure) is subjected to a force P, and supported by a steel tie DC having a circular cross-section 20 mm in diameter. Determine the maximum permissible load P and the deflection of the point B. The allowable stress for the material of the rod CD is 1600 kg/cm².

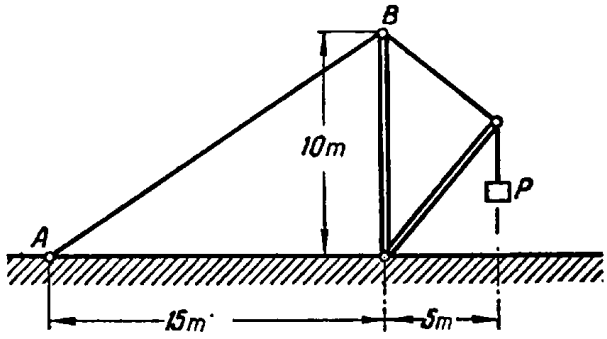
Answer: $P = 1{\cdot}2$ t; $\delta_B = 4{\cdot}17$ mm.



PROBLEM 1.30

1.31. The guy AB of a luffing crane (see figure) is a cable with a cross-section of 500 mm². The allowable stress for the material of the cable is 400 kg/cm². What load P can be raised by the crane, if this depends on the strength of the cable?

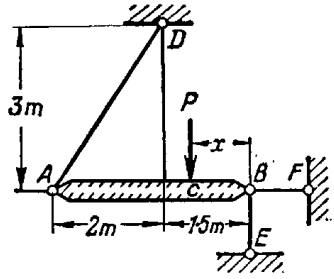
Answer: 3·33 t.



PROBLEM 1.31

1.32. A rigid beam AB is fixed as shown in the figure. The tie AD is of steel, having a circular cross-section 25 mm in diameter; the upright BE is of wood, 1 m long, and has a square cross-section 20 × 20 cm. At what distance x from the support B must the force P be placed if the vertical deflection of the point B is to be twice that of the point A? What must be the force P if the vertical deflection of the point C is to equal 1 mm? What will then be the stresses in the tie AD and the upright BE? The deformation of the horizontal rod BF is to be neglected.

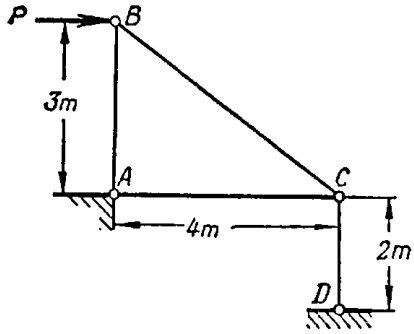
Answer: $x = 0{\cdot}081$ m; $P = 41{\cdot}4$ t; $\sigma_{AD} = 234$ kg/cm²; $\sigma_{BE} = -101$ kg/cm².



PROBLEM 1.32

1.33. In the structure, shown diagrammatically in the figure, all the rods are of steel, having identical cross-sectional areas of 30 cm² each. The force P is 10 t. Determine the stresses in the rods.

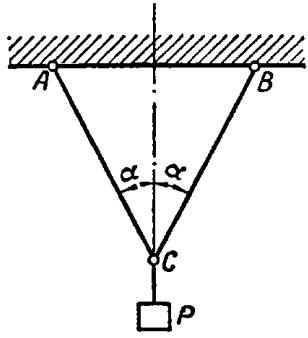
Answer: $\sigma_{AB} = 250$ kg/cm^2; $\sigma_{AC} = 333$ kg/cm^2; $\sigma_{BC} = -417$ kg/cm^2; $\sigma_{CD} = -250$ kg/cm^2.



PROBLEM 1.33

1.34. A load P is suspended from two rods as indicated in the figure. The angle α is 30°. The rod AC is of steel, having a circular cross-section 30 mm in diameter, and an allowable stress $[\sigma]_s$ of 1600 kg/cm^2; the rod CB is of aluminium, having a diameter of 40 mm, and an allowable stress $[\sigma]_a$ of 600 kg/cm^2. What is the maximum load P which can be suspended from these rods?

Answer: 13 t.

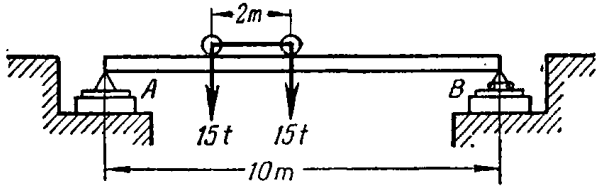


PROBLEM 1.34

1.35. A rigid beam rests on supports A and B. Two connected loads 15 t each (see figure) move along the beam. The cross-section of the supports is square viewed in plan. By considering the most unfavourable position of the loads on the beam, determine the support

dimensions in plan. The allowable stress for the masonry under the supports is 10 kg/cm².

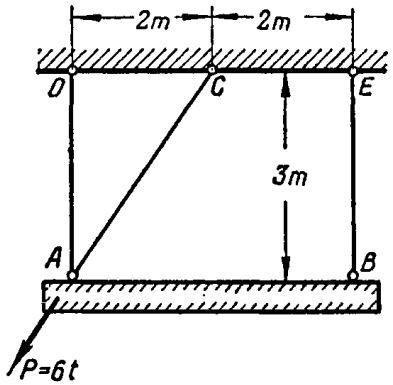
Answer: 52 × 52 cm².



PROBLEM 1.35

1.36. A rigid bar AB is suspended from three steel rods with identical cross-sectional areas F of 10 cm² each (see figure). Determine the stress in the rods, and also the vertical and horizontal displacements of the point A.

Answer: $\sigma_{AD} = \sigma_{BE} = 0$; $\sigma_{AC} = 600$ kg/cm²; $\Delta_{vert.} = 0$; $\Delta_{hor.} = 1{\cdot}95$ mm.



PROBLEM 1.36

1.37. A steel rod of circular cross-section is subjected to a tensile force of 10 t. The strain must not exceed 1/2000, and the stress must not be more than 1200 kg/cm². Find the smallest diameter of the rod to satisfy these conditions.

Solution. We shall express the strain in terms of the load P, the area of the cross-section $\pi d^2/4$, and the modulus of elasticity E—

and shall equate it with the given permissible strain $[\varepsilon]$:

$$\frac{P}{\frac{\pi d^2}{4} E} \leq [\varepsilon].$$

Hence we shall find the required diameter satisfying the given condition:

$$d = 2\sqrt{\frac{P}{\pi E[\varepsilon]}} = 2\sqrt{\frac{10^4 \times 2 \times 10^3}{3{\cdot}14 \times 2 \times 10^6}} = 3{\cdot}57 \text{ cm}.$$

Now, independently from the preceding solution, we shall express the limiting stress condition in terms of P, $\pi d^2/4$, and $[\sigma]$:

$$\frac{P}{\frac{\pi d^2}{4}} \leq [\sigma],$$

whence

$$d = 2\sqrt{\frac{P}{\pi[\sigma]}} = 2\sqrt{\frac{10^4}{3{\cdot}14 \times 1200}} = 3{\cdot}25 \text{ cm}.$$

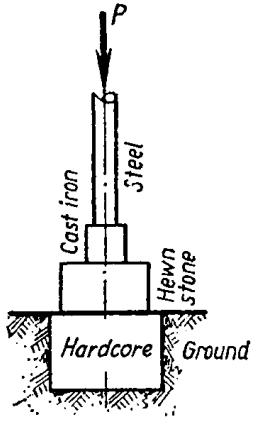
From the two diameters calculated, the larger one, i.e. 3·57 cm, must be taken, since the smaller diameter does not satisfy the required condition of rigidity.

1.38. Select the square cross-sections of the individual parts of the column indicated in the figure, if $P = 100$ t, and the allowable compressive stresses are as follows:

for steel	$[\sigma_s]$	$= 1400$ kg/cm^2
for cast iron	$[\sigma_{c.i.}]$	$= 1000$ kg/cm^2
for hewn stone	$[\sigma_{st}]$	$= 40$ kg/cm^2
for rubble	$[\sigma_r]$	$= 15$ kg/cm^2
for ground (sandy)	$[\sigma_g]$	$= 5$ kg/cm^2

The weight of the parts of the column should not be taken into account.

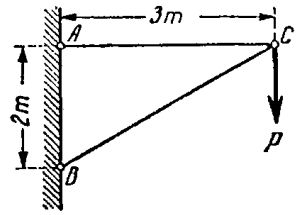
Answer: The cross-section of the steel column: 10×10 cm^2; the cross-section of the cast iron block: 50×50 cm^2; the cross-section of the hewn stone base: 82×82 cm^2; the cross-section of the rubble foundation: 142×142 cm^2.



PROBLEM 1.38

1.39. A bracket (see figure) is subjected to a force P of 6 t. The bar AC is of steel, with an allowable stress for the material $[\sigma]_s$ of 1600 kg/cm^2; the bar BC is of wood with an allowable stress $[\sigma]_w$ of 40 kg/cm^2. Choose the diameter of the circular cross-section of the steel bar, and the dimensions of the square cross-section of the wooden bar, then determine the horizontal, the vertical and the total displacement of the joint C.

Answer: $d = 27$ mm; $a = 16{\cdot}4$ cm; $\delta_{\text{hor.}} = 1{\cdot}4$ mm; $\delta_{\text{vert.}} = 6{\cdot}2$ mm; $\delta = 6{\cdot}64$ mm.

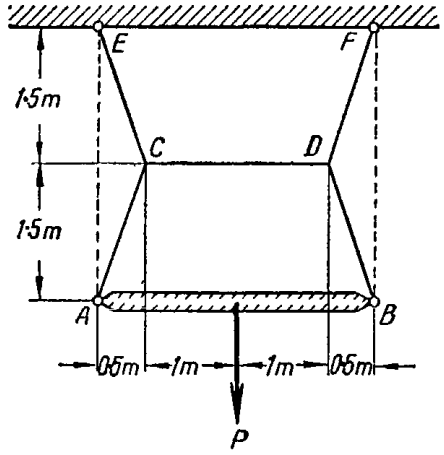


PROBLEM 1.39

1.40. A rigid bar is suspended, as shown in the figure, from a system of hemp ropes. Determine the maximum possible load P, if the allowable stress for the material is assumed to be 100 kg/cm^2;

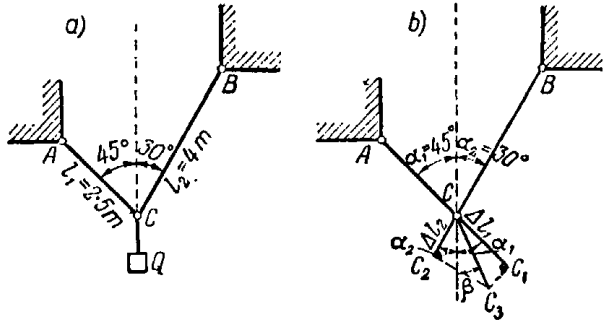
all the ropes are of the same cross-section with the diameter $d = 25$ mm. The useful area of the cross-section constitutes 75% of the area contained by the circumference of the rope.

Answer: 695 kg.



PROBLEM 1.40

1.41. Two rods AC and BC (see figure (a)) are fastened to the supports by pinned joints, and joined together at the point C by a pin from which a load Q (= 5 t) is suspended. The rod AC is made of duralumin, whereas the rod BC is of steel. The allowable stress $[\sigma]$ for the materials of both rods is the same, and is 1500 kg/cm². Find the required cross-sectional areas of the rods, and also the vertical and horizontal displacements of the point C.



PROBLEM 1.41

Answer: $F_{AC} = 1{\cdot}73$ cm²; $F_{BC} = 2{\cdot}44$ cm²; $\Delta_{vert.} = 5$ mm; $\Delta_{hor.} = 2{\cdot}6$ mm.

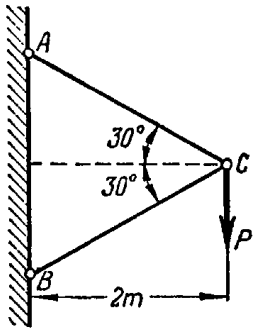
Instruction. To determine the displacement of the point C we shall disconnect the rods at the joint, and represent their new lengths by AC_1 and BC_2, while increasing the original lengths by $\Delta l_1 = CC_1$ and $\Delta l_2 = CC_2$ (see figure (b)). To find the new position of the point C we shall bring together these lengthened rods, rotating them about the points A and B. The points C_1 and C_2 will be displaced along the arcs C_1C_3 and C_2C_3 which, because they are small, may be taken as straight lines perpendicular to AC_1 and BC_2. The length CC_3 will then be the required displacement Δ. Designating by β the angle formed by this movement and the vertical, we obtain the system of equations:

$$\Delta = \frac{\Delta l_1}{\cos(\alpha_1 - \beta)} = \frac{\Delta l_2}{\cos(\alpha_2 + \beta)}.$$

From these equations we determine β, then Δ, and its vertical and horizontal components.

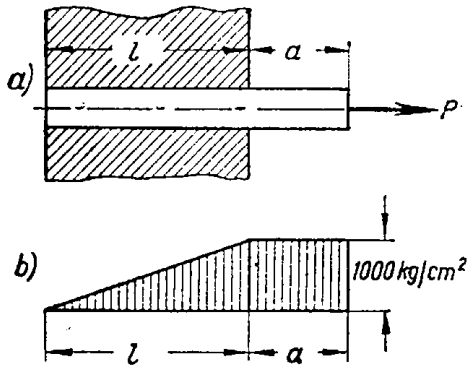
1.42. A bracket ABC (see figure) consists of two bars AC and BC. It is subjected to a force P. The bar AC is of steel—made up of two No. 10 channels (see Appendix A); the bar BC is also of steel—consisting of one No. 20a I-section. The allowable stress for the bar AC is 1600 kg/cm², whereas for the bar BC it is 1000 kg/cm². Determine the maximum permissible load P, and the vertical displacement of the joint C.

Answer: $P = 35{\cdot}5$ t; $\delta = 2{\cdot}76$ mm.



PROBLEM 1.42

1.43. An iron rod, cast in concrete, is held in it by the bonding forces uniformly distributed along its length. To extract the rod a force P of 2 t is applied to one of its ends (see figure (a)). The cross-sectional area F of the rod is 2 cm^2; the length l is 40 cm, the length a is 15 cm. Construct a diagram showing the variation of stress in the different sections along the length of the rod. Determine also its elongation.



PROBLEM 1.43

Solution. In any cross-section of the portion of the rod of length a the stresses are the same, and are given by

$$\sigma = \frac{P}{F} = \frac{2000}{2} = 1000 \text{ kg/cm}^2.$$

In the portion of the rod of length l, for equilibrium, the resultant of the bond forces must be equal to the force P. According to the statement of the problem these forces are distributed uniformly along the length l. Consequently, a unit of this length is subjected to a force p, given by

$$p = \frac{P}{l}.$$

We shall take a section at a distance x from the left-hand end of the rod. Then the resultant of forces on the left of this section will be

$$S = px = \frac{P}{l} x.$$

Having discarded the right-hand portion of the rod we find, from the conditions of equilibrium, that the force in the section is S. Then the stresses in a cross-section at a distance x from the left-hand end of the rod will be

$$\sigma(x) = \frac{S}{F} = \frac{Px}{lF}.$$

At the left-hand end of the rod, at $x = 0$, they are equal to zero, whereas at the distance l from the left-hand end they are

$$\sigma_l = \frac{Pl}{lF} = \frac{P}{F} = 1000 \text{ kg/cm}^2.$$

The diagram showing the variation of stresses along the length of the rod is given in the figure (b), (a linear law).

The elongation of the portion of the rod of length a will equal

$$\Delta l_a = \frac{Pa}{EF} = \frac{2000 \times 15}{2 \times 10^6 \times 2} = 7{\cdot}5 \times 10^{-3} \text{ cm}.$$

For the determination of the elongation of the portion of the rod of length l we shall take, at a distance x from its left-hand end, an infinitely small part of the rod having the length $\mathrm{d}x$. The force in this section was determined earlier. Then the elongation of this infinitely small part of the length of the rod will be

$$\mathrm{d}\,\Delta l = \frac{S\,\mathrm{d}x}{EF} = \frac{Px\,\mathrm{d}x}{lEF}.$$

The elongation of the whole portion of length l will be obtained by the integration:

$$\Delta l_l = \int_0^l \mathrm{d}\,\Delta l = \int_0^l \frac{Px\,\mathrm{d}x}{lEF} = \frac{Pl}{2EF} = \frac{2000 \times 40}{2 \times 2 \times 10^6 \times 2} =$$

$$= 1 \times 10^{-2} \text{ cm}.$$

The total elongation of the rod will be

$$\Delta l = \Delta l_l + \Delta l_a = (1 \times 10^{-2}) + (7{\cdot}5 \times 10^{-3}) =$$
$$= 17{\cdot}5 \times 10^{-3} \text{ cm} = 0{\cdot}175 \text{ mm}.$$

1.44. A load is suspended from a steel wire, the dimensions of which prior to the deformation were as follows: $l = 3$ m and $d = 1{\cdot}6$ mm. The elongation of the wire was found to be equal to 1·5 mm. The same load was then suspended from a copper wire, with the length $l_1 = 1{\cdot}8$ m and diameter $d_1 = 3{\cdot}2$ mm. Its elongation was found to be 0·39 mm. Determine the modulus of elasticity of the copper wire, if the modulus of elasticity of the steel wire E is 2×10^6 kg/cm².

Answer: $1{\cdot}15 \times 10^6$ kg/cm².

1.45. A mine cage weighing 100 kg is suspended from a cable having a diameter d of 10 mm. The length of the cable, when loaded only by the weight of the cage, is 100 m; its length when the cage is loaded with 400 kg of ore is 3 cm greater. Determine the modulus of elasticity of the cable.

Answer: $1{\cdot}7 \times 10^6$ kg/cm².

1.46. Two lines 25 cm apart were drawn on a steel rod 2·5 cm in diameter. When the rod was subjected to a pull of 10,000 kg, the distance between these lines became 25·027 cm. What will be the distance between the lines, if (a) the pull equals 5 t, (b) the tensile stress equals 800 kg/cm? What is the modulus of elasticity E of the material?

Answer: In case (a) 25·0135 cm; in case (b) 25·011 cm; the modulus $E = 1{\cdot}88 \times 10^6$ kg/cm².

1.47. A steel rod of circular cross-section ($d = 32$ mm and $l = 35$ cm) was subjected to a tensile force of 13·5 t in a testing machine. A reduction in diameter of 0·062 mm, and an elongation in a 5 mm length of 0·040 mm, were measured. Determine the modulus of elasticity and Poisson's ratio for the material.

Answer: $E = 2{\cdot}1 \times 10^6$ kg/cm²; $\mu = 0{\cdot}242$.

1.48. A steel rod 6 m long is pulled by a force of 20 t; the modulus of elasticity of the material (E) is 2×10^6 kg/cm^2, and Poisson's ratio (μ) is 0·25. Determine the increase in the volume of the rod.

Answer: 3 cm^3.

1.49. While testing (with a test load) the diagonal member of a steel truss it was found that the difference of the extensometer readings was 10 mm. The gauge length of the extensometer (the length over which the measurement of deformation is carried out) is 20 mm; its magnification factor is 1000. What were the stresses in the diagonal?

Solution. We determine the stresses by Hooke's law. The actual deformation is the difference of the extensometer readings divided by the magnification factor:

$$\Delta l = \frac{\Delta}{k} = \frac{10}{1000} = 0{\cdot}01 \text{ mm}.$$

The strain is equal to the ratio of the deformation to the gauge length of the instrument:

$$\varepsilon = \frac{\Delta l}{l} = \frac{0{\cdot}01}{20} = 0{\cdot}0005.$$

The stress is given by

$$\sigma = E\varepsilon = 2 \times 10^6 \times 0{\cdot}0005 = 1000 \text{ kg/cm}^2.$$

1.50. An extensometer with a gauge length of 100 mm and a magnification of 500 is set on to a component working in tension. The transverse cross-section of the component has an area of 10 cm^2. With the application of a load of 10 t the difference in the extensometer readings was found to be 25 mm. What is the modulus of elasticity of the material of this component?

Answer: $E = 2{\cdot}07 \times 10^6$ kg/cm^2.

1.51. What gauge length must be used for an extensometer intended for testing a steel column, if its magnification factor is 1000,

the reading accuracy does not exceed 0·1 mm, and the stress must be measured with an accuracy not less than 10 kg/cm²?

Answer: $l = 20$ mm.

1.52. While testing a steel rod 2·5 cm in diameter the following results were obtained (see Table A).

The deformations were measured over a gauge length of 20 cm by means of a mirror extensometer giving a magnification of 500. Determine the limit of proportionality, the modulus of elasticity, and draw the stress–strain diagram.

TABLE A

Load t	Extensometer reading mm
0	28·0
1	38·0
2	48·8
3	58·2
4	69·2
5	79·3
6	89·4
7	99·2
8	110·0
9	119·4
9·2	121·4
9·4	123·8
9·6	127·0
9·8	132·8

TABLE B

Increment of load t	Increment of reading mm
1	10·0
1	10·8
1	9·4
1	11·0
1	10·1
1	10·1
1	9·8
1	10·8
1	9·4
0·2	2·0
0·2	2·4
0·2	3·2
0·2	5·8

Solution. We shall compile a table of the increments of the extensometer readings for each step in loading (see Table B). We see from this table that proportionality between load and deformation no longer holds after the value of load has reached 9·2 t. Consequently, the stress at the limit of proportionality is

$$\sigma_p = \frac{P_p}{F} = \frac{9200}{3{\cdot}14 \times 1{\cdot}25^2} = 1870 \text{ kg/cm}^2.$$

From zero load to the load P_p corresponding to the limit of proportionality the total difference in the extensometer readings is

$$\Delta = 121{\cdot}4 - 28{\cdot}0 = 93{\cdot}4 \text{ mm}.$$

The corresponding elongation is

$$\Delta l = \frac{\Delta}{k} = \frac{93{\cdot}4}{500} = 186{\cdot}8 \times 10^{-3} \text{ mm} = 186{\cdot}8 \times 10^{-4} \text{ cm},$$

whereas the strain is

$$\varepsilon = \frac{\Delta l}{l} = \frac{186{\cdot}8 \times 10^{-4}}{20} = 93{\cdot}4 \times 10^{-5}.$$

The modulus of elasticity

$$E = \frac{\sigma_p}{\varepsilon} = \frac{1870}{93{\cdot}4 \times 10^{-5}} = 2{\cdot}0 \times 10^6 \text{ kg/cm}^2.$$

The construction of the stress–strain diagram from the data of the table is not difficult, and has not been included.

1.53. A concrete prism having a square base with 10 cm sides was tested in compression; the following results were obtained:

Load, kg	400	1300	2300	2800	4000	4500
Contraction, mm (over 20 cm length)	0·000386	0·00139	0·00250	0·00312	0·00450	0·00490

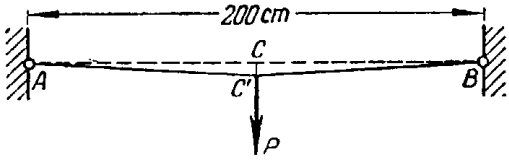
Construct the stress–strain curve, and determine the modulus E for a stress of 40 kg/cm^2.

Answer: $E = 1{\cdot}78 \times 10^5$ kg/cm^2.

Note. The relationship between σ and ε will not be strictly linear. To determine E at the stress of 40 kg/cm^2 it is necessary to assume the curve to be a straight line from zero to this value.

1.54. A steel wire 1 mm in diameter is stretched horizontally between fixed points A and B (see figure). What vertical force P

must be applied at the point C (at mid-span of the wire), if the vertical displacement of the point C is to be 4·5 cm? What stress will then occur in the wire? Neglect the weight of the wire.



PROBLEM 1.54

Solution. The elongation of one end of the wire, after applying the force P, will be equal to the difference between the lengths CB and C'B:

$$\Delta l = \sqrt{100^2 + 4{\cdot}5^2} - 100 = 0{\cdot}1 \text{ cm}.$$

The strain

$$\varepsilon = \frac{0{\cdot}1}{100} = 0{\cdot}001.$$

We can now determine the stress in the wire by applying Hooke's law:

$$\sigma = E\varepsilon = 2 \times 10^6 \times 0{\cdot}001 = 2000 \text{ kg/cm}^2.$$

Knowing the stress, we determine the force in the wire:

$$S = \sigma F = 2000 \frac{3{\cdot}14 \times 0{\cdot}1^2}{4} = 15{\cdot}7 \text{ kg}.$$

The point C' will be in equilibrium under the action of the force P and the two forces S. From the similarity of the triangle of forces and the triangle C'CB we find that

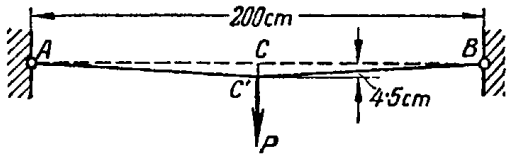
$$\frac{(P/2)}{4{\cdot}5} = \frac{S}{100{\cdot}1},$$

whence

$$P = 2 \times 4{\cdot}5 \frac{S}{100{\cdot}1} = \frac{9 \times 15{\cdot}7}{100{\cdot}1} = 1{\cdot}41 \text{ kg}.$$

1.55. A steel wire 1 mm in diameter was stretched horizontally between fixed points A and B (see figure). From the point C (at mid-span of the wire) a gradually increasing load P was applied. The wire broke when its elongation reached 0·5%. What was the load P at that instant? What was the vertical displacement of the point C, and the stress in the wire at the instant of breaking? Neglect the weight of the wire. Assume that the wire is taut and has only elastic deformations up to the instant of the break.

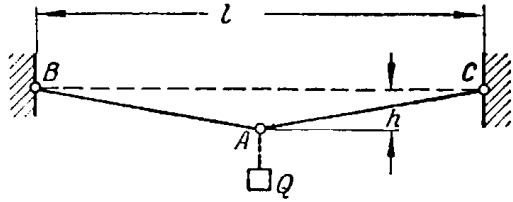
Answer: $P = 15{\cdot}7$ kg; $CC' = 10$ cm; $\sigma = 100$ kg/cm^2.



PROBLEM 1.55

1.56. A load Q of 160 kg is suspended (see figure) from two steel wires AB and AC each having the same length; the distance l is $5h$. Determine the required diameter of the wires, and the vertical displacement of the point A after applying the load. Determine this diameter also for $l = 10h$. Neglect the weight of the wire in both cases.

Answer: 5·25 mm; 7·21 mm; $\Delta h = 0{\cdot}0007l$.



PROBLEM 1.56

1.57. Determine the factor of safety for the material of the piston rod in Problem 1.1 with respect to the ultimate stress σ_u, the yield stress σ_y, and the limit of endurance $\sigma_{end.}$, taking, for the steel of which the rod is made, the following values:

$\sigma_u = 50$ kg/mm^2, $\sigma_y = 30$ kg/mm^2 and $\sigma_{end} = 14$ kg/mm^2.

Answer: $k_u = 10$; $k_y = 6$; $k_{end.} = 2{\cdot}8$.

1.58. What must be the allowable stress for the material of a railway waggon coupling when calculated on a static load, if during a sudden start of the train the stress is not to exceed the yield point of the steel? We take the yield point as $\sigma_y = 0{\cdot}6\sigma_u$, and the ultimate strength $\sigma_u = 50$ kg/mm². Allow for the stresses during a sudden start to be twice those occurring during a gradual start.

Answer: 1500 kg/cm².

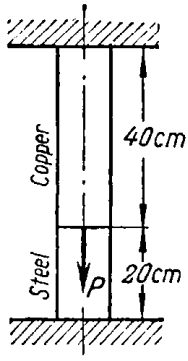
1.59. Determine the area of the cross-section of a steel rod subjected to a suddenly applied tensile force $P = 1{\cdot}5$ t having an impact factor k_d of 2; the strain during this period must not exceed 1/1800. Determine the factor of safety if the ultimate strength of the material σ_u is 42 kg/mm².

Answer: $F = 2{\cdot}7$ cm²; $k = 3{\cdot}78\,k_d$.

2. STATICALLY INDETERMINATE SYSTEMS

1.60. A rod with the upper part of copper and the lower part of steel (see figure) is subjected to a force P of 10 t. Both ends of the rod are rigidly clamped. The area of its cross-section F is 20 cm². Determine the stress in each part of the rod.

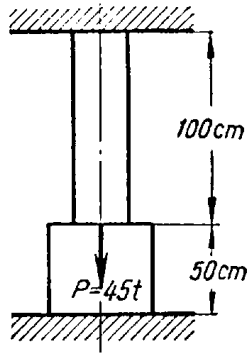
Answer: $\sigma_c = 100$ kg/cm²; $\sigma_s = -400$ kg/cm².



PROBLEM 1.60

1.61. A rod rigidly clamped at both ends (see figure) has a cross-sectional area of 10 cm² for its upper part, and 40 cm² for its lower part. Determine the stresses in each part of the rod.

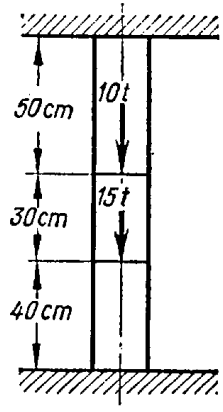
Answer: $\sigma_u = 500$ kg/cm²; $\sigma_l = -1000$ kg/cm².



PROBLEM 1.61

1.62. A rod with a cross-sectional area F of 10 cm² is rigidly clamped at both ends and loaded as shown in the figure. Determine the stresses in all three parts of the rod.

Answer: $\sigma_u = 1083$ kg/cm²; $\sigma_m = 83$ kg/cm²; $\sigma_l = -1417$ kg/cm².



PROBLEM 1.62

1.63. A hawser consists of a central copper wire with a diameter d_1 of 5 mm, and nine steel wires each with a diameter d_2 of 2·5 mm,

surrounding the former. Find the stresses in the hawser due to a load P of 500 kg, assuming the wires to remain parallel.

Answer: $\sigma_c = 4{\cdot}63$ kg/mm^2; $\sigma_s = 9{\cdot}26$ kg/mm^2.

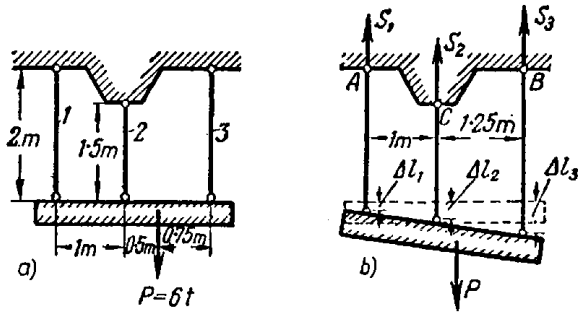
1.64. The cross-sectional area of the concrete in a short reinforced concrete column is 645 cm^2. The column is provided with four longitudinal steel rods arranged symmetrically, each having a cross-sectional area of 10 cm^2. Determine the allowable load on the column, if $[\sigma_c]$ is 80 kg/cm^2 and $[\sigma_s]$ is 1400 kg/cm^2. The modulus of elasticity for the steel is 2×10^6 kg/cm^2, whereas for the concrete it is 2×10^5 kg/cm^2.

Answer: 83·6 t.

1.65. A concrete column of square cross-section is reinforced by four steel rods, the cross-sectional area of which forms 1% of the cross-sectional area of the column. The allowable stress for the concrete is 60 kg/cm^2; for the reinforcement it is 1200 kg/cm^2. The ratio of the modulus of elasticity of the steel to that of concrete is 10. The column carries a load of 100 t. What must be the side of the cross-section of the column, and the diameter of the rods?

Answer: $a = 39$ cm; $d = 22$ mm.

1.66. A rigid bar, the deformations of which may be neglected, is suspended from three rods and loaded as shown in the figure (a). Rod 1 is of copper, with a cross-sectional area of 1 cm^2; rod 2 is of steel, with a cross-sectional area of 1·5 cm^2; rod 3 is of aluminium, with a cross-sectional area of 2 cm^2. Determine the stress in the rods.



PROBLEM 1.66

Solution. We assume that all the forces are tensile and that the beam, after the deformation of the rods, will take up the new position shown in the figure (b).

We shall set up the two equations of equilibrium:

$$\sum m_A = -(S_2 \times 1) - (S_3 \times 2{\cdot}25) + (P \times 1{\cdot}5) = 0, \qquad (1)$$

$$\sum m_B = (S_1 \times 2{\cdot}25) + (S_2 \times 1{\cdot}25) - (P \times 0{\cdot}75) = 0. \qquad (2)$$

For the setting up of the necessary third equation we shall consider the deformations (figure (b)). From the trapezium in the figure it is possible to set up the following relationship:

$$\frac{\Delta l_3 - \Delta l_1}{2{\cdot}25} = \frac{\Delta l_2 - \Delta l_1}{1},$$

which we shall re-arrange as

$$1{\cdot}25\,\Delta l_1 - 2{\cdot}25\,\Delta l_2 + \Delta l_3 = 0.$$

We shall express the deformation in terms of the forces:

$$1{\cdot}25\,\frac{S_1 \times 200}{10^6 \times 1} - 2{\cdot}25\,\frac{S_2 \times 150}{2 \times 10^6 \times 1{\cdot}5} + \frac{S_3 \times 200}{0{\cdot}7 \times 10^6 \times 2} = 0.$$

After cancelling out we have

$$(2{\cdot}5 \times S_1) - (1{\cdot}125 \times S_2) + (1{\cdot}43 \times S_3) = 0. \qquad (3)$$

Solving the simultaneous equations (1), (2) and (3) we obtain:

$$S_1 = 0{\cdot}018P = 0{\cdot}018 \times 6000 = \ 110 \text{ kg},$$

$$S_2 = 0{\cdot}568P = 0{\cdot}568 \times 6000 = 3410 \text{ kg},$$

$$S_3 = 0{\cdot}414P = 0{\cdot}414 \times 6000 = 2480 \text{ kg}.$$

We now check the values obtained:

$$\sum S = 110 + 3410 + 2480 = 6000 \text{ kg} = P.$$

The stresses in the rods are respectively:

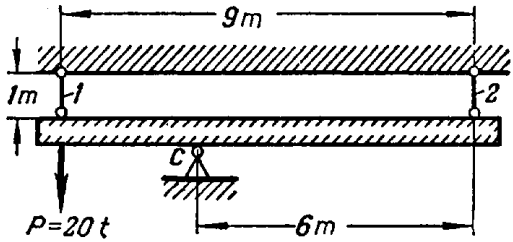
$$\sigma_1 = \frac{S_1}{F_1} = \frac{110}{1} = 110\,\text{kg/cm}^2,$$

$$\sigma_2 = \frac{S_2}{F_2} = \frac{3410}{1{\cdot}5} = 2270\,\text{kg/cm}^2,$$

$$\sigma_3 = \frac{S_3}{F_3} = \frac{2480}{2} = 1240\,\text{kg/cm}^2.$$

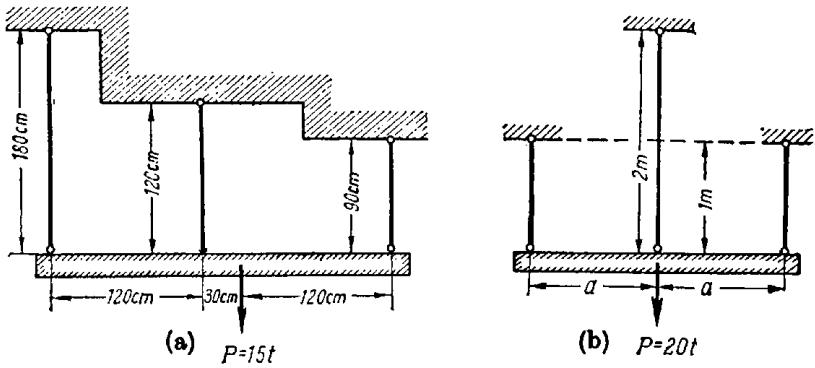
1.67. Determine the magnitude of the stresses occurring in the rods 1 and 2 supporting a rigid bar (see figure). Rod 1 is of copper, with a cross-sectional area of 20 cm²; rod 2 is of steel, with a cross-sectional area of 10 cm².

Answer: $\sigma_1 = 200\,\text{kg/cm}^2$; $\sigma_2 = -800\,\text{kg/cm}^2$.

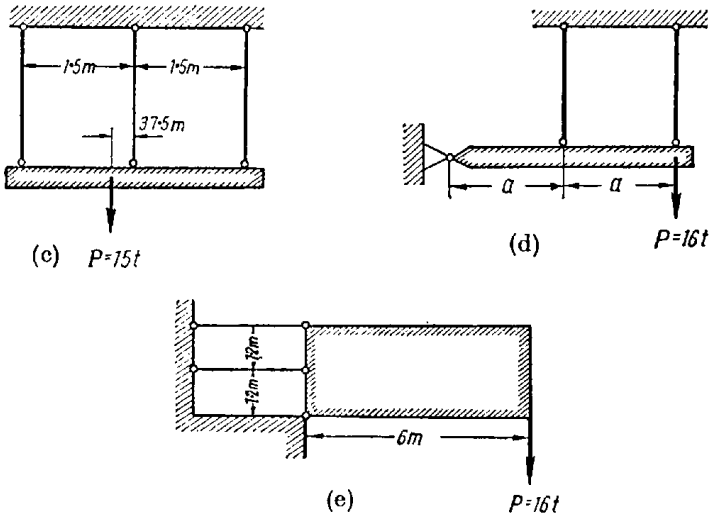


PROBLEM 1.67

1.68. A rigid bar is fastened by a system of steel rods of identical cross-section (see figure). Assuming that $[\sigma]$ is 1600 kg/cm², determine the cross-sectional area of the rods.



PROBLEM 1.68

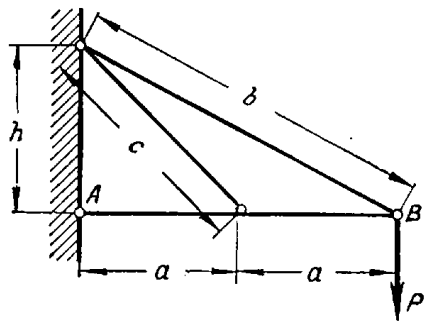


PROBLEM 1.68

Answer: (a) 3·78 cm²; (b) 5 cm²; (c) 4·3 cm²; (d) 8 cm²; (e) 20 cm².

1.69. A rigid bar AB (the deformation of which is neglected) is fastened by a hinge to a wall at point A, and suspended from two wires of identical material and cross-sectional area, as shown in the figure. Show that the forces in the wires, due to load P applied at point B, are

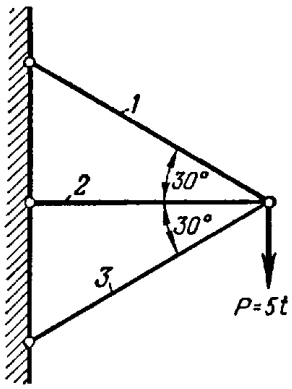
$$\frac{2P}{h} \times \frac{cb^3}{b^3 + 4c^3} \quad \text{and} \quad \frac{4P}{h} \times \frac{bc^3}{b^3 + 4c^3}.$$



PROBLEM 1.69

1.70. The three bars of a bracket (see figure) are made of the same material. The cross-sectional areas of the first, second, and third bar are 2 cm^2, 3 cm^2, and 4 cm^2 respectively. Determine the stresses in the bars.

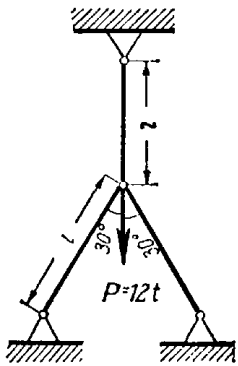
Answer: $\sigma_1 = 2113\,kg/cm^2$; $\sigma_2 = 447\,kg/cm^2$; $\sigma_3 = -1440\,kg/cm^2$.



PROBLEM 1.70

1.71. Three rods pin-jointed together at a point (see figure) are of identical cross-section. Determine the area of the cross-section, assuming that $[\sigma]$ is 1600 kg/cm^2.

Answer: 3 cm^2.

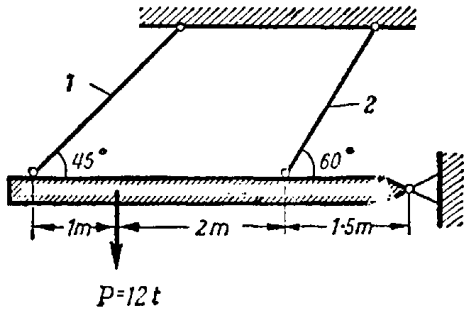


PROBLEM 1.71

1.72. A rigid beam is suspended from two hangers as shown in the figure. The first hanger must have a cross-sectional area twice that

of the second one. The material of the hangers is steel, and the allowable stress $[\sigma]$ is 1600 kg/cm^2. Determine the cross-sectional areas of the hangers.

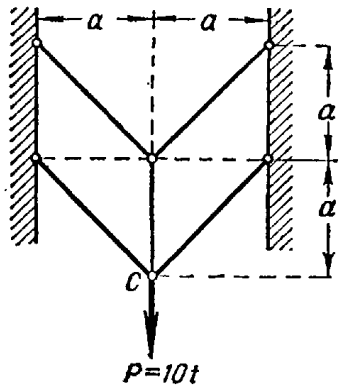
Answer: $F_1 = 7{\cdot}5$ cm^2; $F_2 = 3{\cdot}75$ cm^2.



PROBLEM 1.72

1.73. Determine the vertical displacement of point C of a system of steel rods (see figure) each having a cross-sectional area of 5 cm^2. The dimension a is 1 m.

Answer: 0·895 mm.

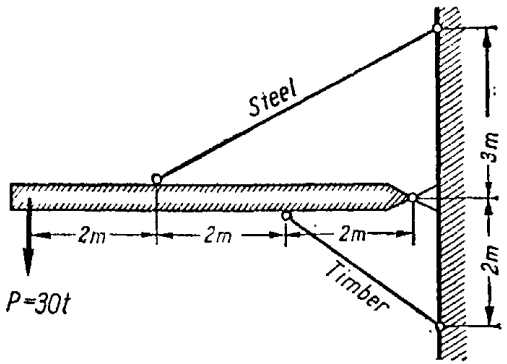


PROBLEM 1.73

1.74. A rigid bar is fastened to a wall by means of a hinge (see figure), a steel tie and a timber strut. Find the cross-sectional areas of the tie and strut, if the cross-sectional area of the strut is to be

10 times that of the tie, and the allowable stresses are 1600 kg/cm² for steel, and 60 kg/cm² for wood.

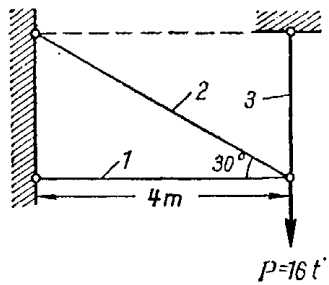
Answer: $F_{\text{tie}} = 50$ cm²; $F_{\text{strut}} = 500$ cm².



PROBLEM 1.74

1.75. In the structure shown rod 1 is of cast iron, with $[\sigma_{c.i.}]$ 800 kg/cm²; rod 2 is of copper, with $[\sigma_c]$ 600 kg/cm²; rod 3 is of steel, with $[\sigma_s]$ 1200 kg/cm². The cross-sectional areas of rods 1 and 2 are equal; the cross-sectional area of rod 3 is half that of rods 1 and 2. Find the cross-sectional area of the rods.

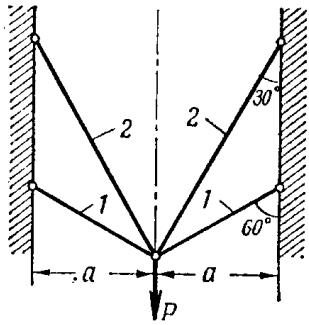
Answer: $F_1 = F_2 = 24{\cdot}6$ cm²; $F_3 = 12{\cdot}3$ cm².



PROBLEM 1.75

1.76. In the structure shown rods 1 are of steel, with a cross-sectional area F_s of 10 cm². Rods 2 are of copper, with a cross-sectional area F_c of 20 cm². The allowable stress for steel $[\sigma_s]$ is 1600 kg/cm², and that for copper $[\sigma_c]$ is 600 kg/cm². Find the cross-sectional area of the rods.

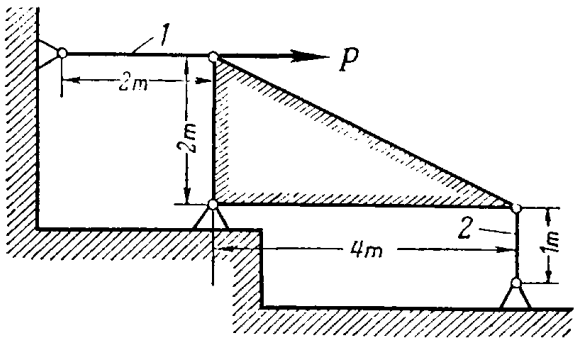
Answer: 32·8 t.



PROBLEM 1.76

1.77. A rigid structure is fastened to a foundation by means of a hinge and two rods (see figure). Rod 1 is of steel ($[\sigma_s] = 1600\,\text{kg/cm}^2$), rod 2 is of cast iron ($[\sigma_{c.i.}] = 1000\,\text{kg/cm}^2$). Their cross-sectional areas F_s and $F_{c.i.}$ are 30 cm² and 50 cm² respectively. Determine the maximum load P.

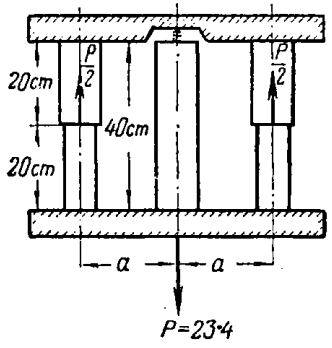
Answer: 112·5 t.



PROBLEM 1.77

1.78. Two rigid bars are joined by three rods as shown in the figure. The end rods are of steel, with a cross-sectional area of the upper part equal to 16 cm², and that of the lower part equal to 10 cm². The middle rod is of copper, with a cross-sectional area equal to 20 cm². A spring with a compliance α of $1{\cdot}25 \times 10^{-6}$ cm/kg is placed between the upper end of the middle rod and the top bar. Determine the stresses in the rods under the given load.

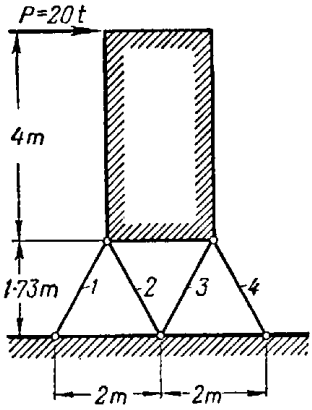
Answer: In the upper part of the steel rods: $\sigma = -90$ kg/cm^2; in the lower part of the steel rods: $\sigma = 1026$ kg/cm^2; in the copper rod: $\sigma = 144$ kg/cm^2.



PROBLEM 1.78

1.79. The four rods supporting a rigid structure (see figure) are made of the same material, and have identical cross-sections with an area F of 25 cm^2 each. Determine the stresses in them.

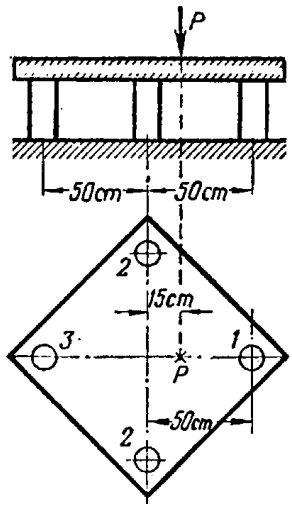
Answer: $\sigma_1 = 1330$ kg/cm^2; $\sigma_2 = 525$ kg/cm^2; $\sigma_3 = -525$ kg/cm^2; $\sigma_4 = -1330$ kg/cm^2.



PROBLEM 1.79

1.80. A square slab is supported on four symmetrically placed uprights of identical cross-section, length and material (see figure). Determine the force in each upright, neglecting the deformation of the slab.

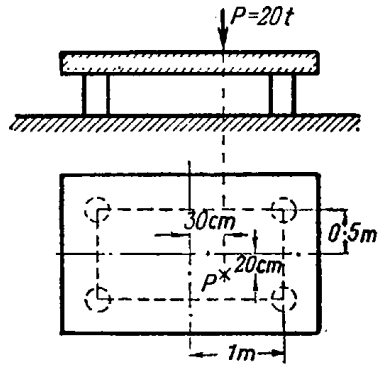
Answer: $S_1 = 0{\cdot}4\,P$; $S_2 = 0{\cdot}25\,P$; $S_3 = 0{\cdot}1\,P$.



PROBLEM 1.80

1.81. A rigid rectangular slab is supported on four uprights of identical cross-section, length and material, placed under its corners as indicated in the figure. Determine the force in each upright.

Answer: $S_1 = 1{\cdot}5$ t; $S_2 = 4{\cdot}5$ t; $S_3 = 8{\cdot}5$ t; $S_4 = 5{\cdot}5$ t.

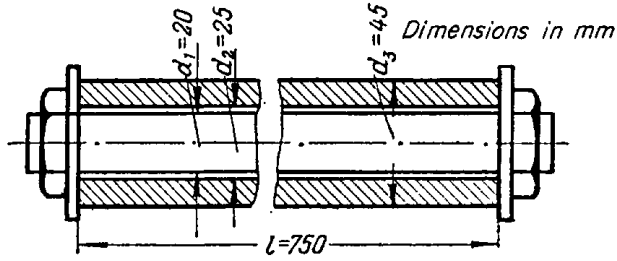


PROBLEM 1.81

1.82. A steel bolt is put through a copper tube as shown in the figure. The pitch of the screw thread of the bolt is 3 mm. What

stresses will occur in the bolt and tube if the nut is turned through one quater of a revolution?

Answer: $\sigma_s = 1274$ kg/cm^2; $\sigma_c = -364$ kg/cm^2.



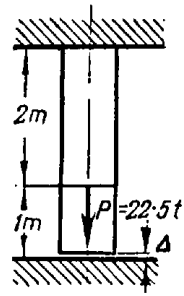
PROBLEM 1.82

1.83. A cast iron cylinder 1·50 m long has an internal diameter of 25 cm and wall thickness of 25 mm. The ends of the cylinder are closed by rigid covers, through the centres of which a steel bolt, tensioned by nuts, is located. The nuts are tightened so that the tensile force in the bolt is 20 t. Then a tensile force of 20 t is applied to each cover. Determine the stress in the bolt if its effective length is 150 cm, and its cross-sectional area is 35 cm.

Answer: 692 kg/cm^2.

1.84. A steel rod (see figure), clamped at its upper end, has a cross-sectional area F of 20 cm^2. Between its lower end and a rigid plane, there was a clearance Δ, of 0·375 mm before the load was applied. Determine the stresses in the upper and lower parts of the rod due to the force P.

Answer: $\sigma_u = 625$ kg/cm^2; $\sigma_l = -500$ kg/cm^2.

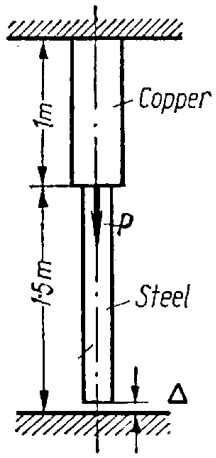


PROBLEM 1.84

1.85. The rod represented in the figure is rigidly clamped at its upper end. Before the load was applied there was a clearance Δ,

of 0·05 mm between its lower end and a rigid support. The upper part, with a cross-sectional area of 150 cm², is of copper; the lower part, with a cross-sectional area of 50 cm², is of steel. Determine the stresses in the upper and lower parts of the rod, occurring after the rod is subjected to a force P of 20 t.

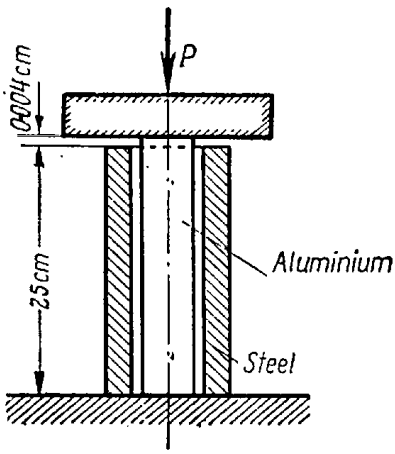
Answer:
$\sigma_u = 107{\cdot}6$ kg/cm²;
$\sigma_l = -77$ kg/cm².



PROBLEM 1.85

1.86. The aluminium rod shown in the figure has a cross-sectional area of 20 cm², and, with zero load, a length of 25·004 cm. The steel tube has the same cross-sectional area, and, under the same conditions, a length of 25 cm. What load, P, gives rise to equal stresses in the steel and the aluminium?

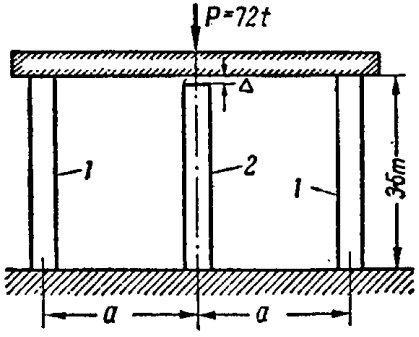
Answer: 6·9 t.



PROBLEM 1.86

1.87. A rigid beam (see figure) is placed on three columns of identical cross-sectional area of 400 cm² each. Before the load was applied there was a clearance Δ, of 1·5 mm between the beam and the middle column. The material of the column is concrete with a modulus E of 140,000 kg/cm². Determine the stresses in the columns.

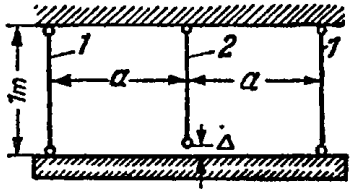
Answer: $\sigma_1 = -80$ kg/cm²; $\sigma_2 = -20$ kg/cm².



PROBLEM 1.87

1.88. A rigid beam (see figure) is suspended from three steel rods. The middle rod is made 0·5 mm shorter than the correct length and assembled with an initial stress. Determine the stresses in the rods, if the cross-sectional area of each of the three rods is 2 cm².

Answer: $\sigma_1 = -333$ kg/cm²; $\sigma_2 = 667$ kg/cm².

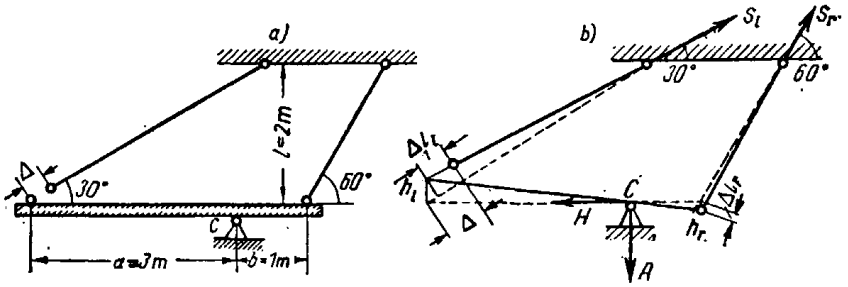


PROBLEM 1.88

1.89. A rigid bar (see figure (a)) is fastened to a foundation by means of hinge C and two steel ties of identical cross-section. The left-hand tie is made 1 mm (= Δ) shorter than the design dimension. Determine the stresses in the ties after the assembly.

Solution. After the assembly of the structure the rigid bar will take up the inclined position indicated in figure (b). Thereby the

left-hand tie will be elongated by an amount Δl_1 and the right-hand tie by an amount Δl_r. At the fixing points of the ties reactions S_1 and S_r will occur; at hinge C reactions A and H will occur. We shall set up the equation of equilibrium: the sum of moments of all forces about hinge C is zero.



PROBLEM 1.89

$$S_1 \times a \sin 30° - S_r \times b \sin 60° = 0,$$

whence

$$S_r = \frac{a \sin 30°}{b \sin 60°} S_1 = \frac{3 \times 0{\cdot}5}{1 \times 0{\cdot}866} S_1 = 1{\cdot}73\, S_1. \qquad \text{(a)}$$

Reactions A and H, which occur in the remaining two equations of equilibrium, do not interest us; we therefore discard them. To obtain an additional equation we shall consider the deformations. From figure (b) it is seen that

$$\frac{h_1}{h_r} = \frac{a}{b} = 3.$$

But

$$h_1 = \frac{\Delta - \Delta l_1}{\sin 30°} \quad \text{and} \quad h_r = \frac{\Delta l_r}{\sin 60°}.$$

Therefore

$$\frac{(\Delta - \Delta l_1) \times \sin 60°}{\sin 30° \times \Delta l_r} = 3$$

or

$$\frac{\Delta - \Delta l_1}{\Delta l_r} = 3 \frac{\sin 30°}{\sin 60°} = 3 \frac{0{\cdot}5}{0{\cdot}866} = 1{\cdot}73,$$

whence

$$1{\cdot}73\, \Delta l_r + \Delta l_1 = \Delta. \qquad \text{(b)}$$

We shall express Δl_1 and Δl_r in terms of forces S_1 and S_r:

$$\Delta l_1 = \frac{S_1 l_1}{EF} = \frac{S_1 l}{EF \sin 30^\circ}$$

and

$$\Delta l_r = \frac{S_r l_r}{EF} = \frac{S_r l}{EF \sin 60^\circ}.$$

Substituting in equation (b) the values we have already found for Δl_1 and Δl_r

$$1{\cdot}73 \frac{S_r l}{EF \sin 60^\circ} + \frac{S_1 l}{EF \sin 30^\circ} = \Delta$$

or

$$S_r + S_1 = \frac{\Delta \times EF}{2l}. \qquad (c)$$

We shall divide both sides of equations (a) and (c) by F, and note that

$$\frac{S_1}{F} = \sigma_1 \quad \text{and} \quad \frac{S_r}{F} = \sigma_r;$$

then

$$\sigma_r = 1{\cdot}73\, \sigma_1, \quad \sigma_r + \sigma_1 = \frac{\Delta \times E}{2l}.$$

Solving these simultaneous equations, we have

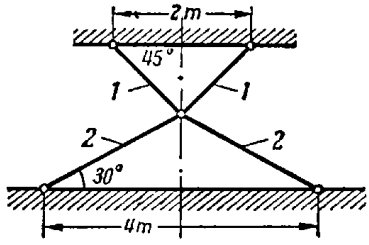
$$\sigma_1 = 0{\cdot}184 \frac{\Delta \times E}{l} = \frac{0{\cdot}184 \times 0{\cdot}1 \times 2 \times 10^6}{200} = 184 \text{ kg/cm}^2,$$

$$\sigma_r = 0{\cdot}317 \frac{\Delta \times E}{l} = \frac{0{\cdot}317 \times 0{\cdot}1 \times 2 \times 10^6}{200} = 317 \text{ kg/cm}^2.$$

1.90. While assembling the structure shown in the figure it was found that the steel rods (denoted by 1) had been made 0·3 mm longer than the design dimension. Their cross-sectional area F_1 is 25 cm² each. The cross-sectional area F_2 of the cast iron rods (de-

noted by 2) is 40 cm² each. Determine the stresses in the rods after the structure had been assembled.

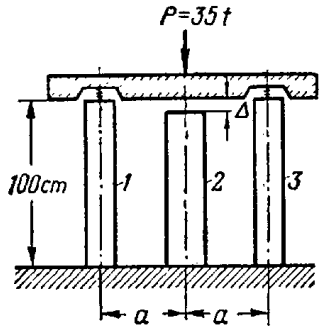
Answer: $\sigma_1 = -96{\cdot}5$ kg/cm²; $\sigma_2 = -85{\cdot}2$ kg/cm².



PROBLEM 1.90

1.91. A rigid bar rests on three vertical supports (see figure). The end supports are of steel, with equal cross-sectional areas of 20 cm² each; the middle support is of cast iron, with a cross-sectional area of 50 cm². Identical springs, with a compliance α equal to 5×10^{-6} cm/kg, are placed between the end supports and the bar. Between the middle support and the bar there was a gap Δ of 0·5 mm before the load was applied. Determine the stresses in the supports.

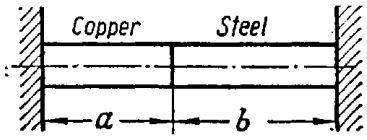
Answer: $\sigma_s = -500$ kg/cm²; $\sigma_{c.i.} = -300$ kg/cm².



PROBLEM 1.91

1.92. A rod of constant cross-section is tightly clamped at both ends. The part with a length (a) of 1 m (see figure) is of copper; the other part with a length (b) of 1·5 m is of steel. After clamping the rod its temperature is increased by 50°C ($= \Delta t$). Determine the stress in the rod.

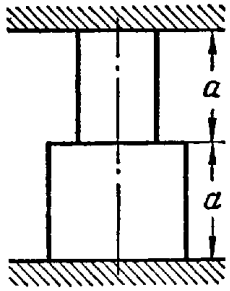
Answer: -1005 kg/cm^2.



PROBLEM 1.92

1.93. A steel bar (see figure) is fixed between two rigid planes at a temperature t_1 of $+5$°C. The cross-sectional area (F_u) of the upper part is 5 cm^2, whereas that (F_l) of the lower part is 10 cm^2. Determine the magnitude of the stresses occurring in each part of the rod, if the temperature is increased to $+25$°C ($= t_2$).

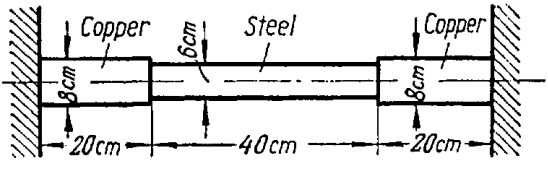
Answer: $\sigma_u = -666$ kg/cm^2; $\sigma_l = -333$ kg/cm^2.



PROBLEM 1.93

1.94. The bar shown in the figure consists of three parts and has a circular cross-section. After clamping the ends of the bar its temperature is increased by 30°C ($= \Delta t$). Determine the stresses in the copper and steel parts of the bar.

Answer: $\sigma_c = -460$ kg/cm^2; $\sigma_s = -820$ kg/cm^2.



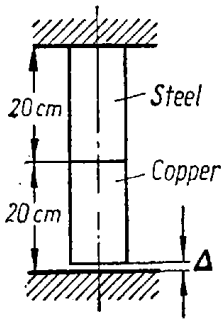
PROBLEM 1.94

1.95. A steel rod of length l (2 m) has its ends fixed in walls. One of the ends can move freely inside the wall over a distance (Δ) of 0·05 mm. Determine the stresses in the rod if the temperature is raised by 40°C.

Answer: -950 kg/cm^2.

1.96. The bar shown in the figure is clamped at the upper end at a temperature t_1 of -15°C; a gap Δ of 0·4 mm is left between the lower end of the bar and a rigid plane. The bar is then heated up to a temperature t_2 of 85°C. What is the stress at this instant?

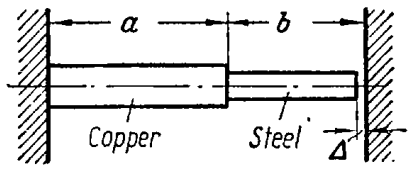
Answer: -600 kg/cm^2.



PROBLEM 1.96

1.97. After the bar shown in the figure was clamped, its temperature was raised by 30°C. Its left-hand part, with a length a of 1·50 m and cross-sectional area of 200 cm^2, is of copper; its right-hand part, with a length b of 1 m and cross-sectional area of 100 cm^2, is of steel. When the bar was clamped there was a gap, Δ, of 0·03 mm between the right-hand end of the bar and the rigid plane. Neglecting the bending deformations, determine the stresses in the two parts of the bar.

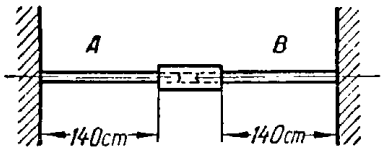
Answer: $\sigma_c = -435$ kg/cm^2; $\sigma_s = -870$ kg/cm^2.



PROBLEM 1.97

1.98. By means of a turnbuckle a tensile stress σ_1 of 150 kg/cm² is set up in steel rods A and B (see figure). By how much should the length of the rods be altered (by means of the turnbuckle), if after a temperature drop of 10°C ($= \Delta t$) the stress σ_2, in them is to be 800 kg/cm²? Neglect the deformation of the turnbuckle.

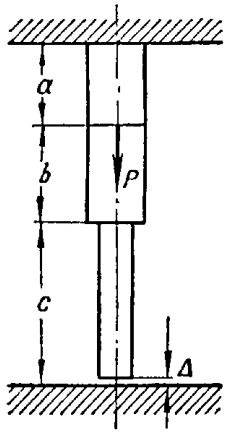
Answer: $\Delta = +0{\cdot}56$ mm (in both rods).



PROBLEM 1.98

1.99. The upper part of the bar shown in the figure is of steel, with a cross-sectional area F_s of 200 cm²; the lower part is of copper, with a cross-sectional area F_c of 100 cm². The gap (Δ) is 0·08 mm, and the dimensions are: $a = 0{\cdot}5$ m, $b = 0{\cdot}5$ m, $c = 1$ m. Determine the stresses in the parts of the bar, due to a force, P, of 150 t and a simultaneous temperature rise of 20°C.

Answer: $\sigma_a = 475$ kg/cm²; $\sigma_b = -275$ kg/cm²; $\sigma_c = -550$ kg/cm².



PROBLEM 1.99

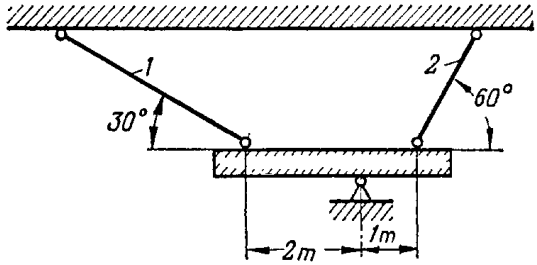
1.100. An invar tube, with a cross-sectional area of 30 cm² and a length of 50 cm, and a copper tube of identical dimensions are

placed end to end at a temperature of 0°C. They are then heated to + 45°C, and compressed by forces preventing any change in their total length. What is the magnitude of these forces, and the length of each tube at the higher temperature? For invar $\alpha_i = 0$ and $E_i = 2 \times 10^6$ kg/cm².

Answer: $P = 15{\cdot}0$ t; $l_c = 50{\cdot}01$ cm; $l_i = 49{\cdot}99$ cm.

1.101. A rigid beam (see figure) is, in addition to a hinged support, suspended from two steel hangers of equal cross-sectional area of 40 cm² each. After the hangers were fixed their temperature was raised by 20°C. Determine the stresses in the hangers.

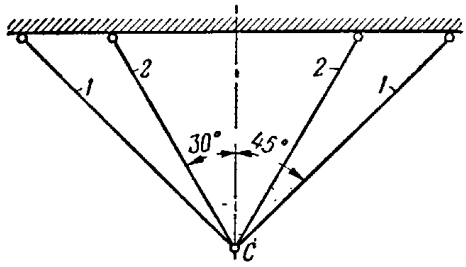
Answer: $\sigma_1 = -470$ kg/cm²; $\sigma_2 = -542$ kg/cm².



PROBLEM 1.101

1.102. Steel rods 1, with a cross-sectional area of 20 cm² each, and copper rods 2, with a cross-sectional area of 30 cm² each, are pin-jointed together at point C (see figure) at a temperature, t_1, of 15°C. Determine the stresses in the rods if the temperature of the structure is reduced to −45°C (= t_2).

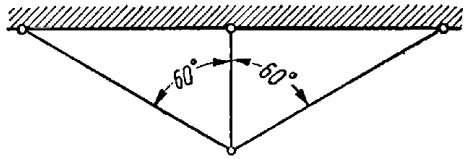
Answer: $\sigma_s = 104$ kg/cm²; $\sigma_c = -57$ kg/cm².



PROBLEM 1.102

1.103. All rods of the structure shown in the figure are of steel and have identical cross-sections. Which is more dangerous: an increase in the temperature of the middle rod only, or an increase in the temperature of the end rods only?

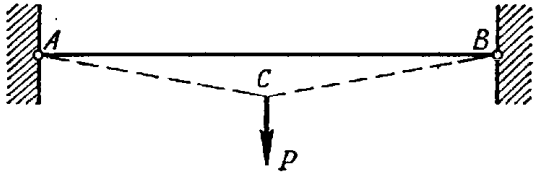
Answer: In the latter case the stresses are four times greater than in the former; thus the latter case is more dangerous.



PROBLEM 1.103

1.104. A wire (the weight of which is neglected) 1 mm in diameter, is suspended horizontally, without tension, between fixed points A and B at a temperature, t_0, of 0°C (see figure). The temperature of the wire can rise to + 50°C (= t_1), and fall to −50°C (= t_2). The allowable stress $[\sigma]$ for the material of the wire is 2000 kg/cm². Determine the load P that may be suspended from point C (at the mid-span) at the three given temperatures.

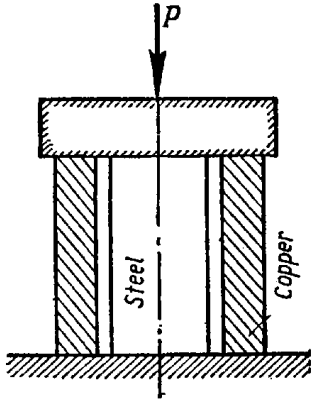
Answer: At 0°C: $P = 1{\cdot}4$ kg; at −50°C: $P = 0{\cdot}32$ kg; at +50°C: $P = 1{\cdot}79$ kg.



PROBLEM 1.104

1.105. By means of a rigid slab a load of 4·5 t is transmitted to a solid steel cylinder with a cross-sectional area of 15 cm², and to a hollow copper cylinder with a cross-sectional area of 20 cm². The forces act along the axes of both cylinders (see figure). After the application of the load the temperature may increase by 30°C. Determine the distribution of the load before and after the change in temperature.

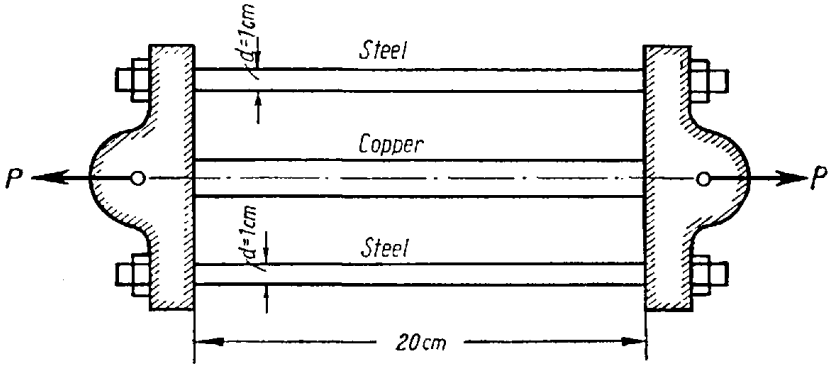
Answer: Before the change in temperature $P_s = 2{\cdot}7\,t$, $P_c = 1{\cdot}8\,t$; after the change in temperature $P_s = 1{\cdot}26\,t$, $P_c = 3{\cdot}24\,t$.



PROBLEM 1.105

1.106. Two rigid castings are fastened by bolts as shown in the figure. Determine the forces P necessary for drawing the castings apart so that a steel rod 20·02 cm long and having a cross-sectional area of 6 cm^2 may be freely inserted between them. Determine the distance between the castings after the removal of the forces P.

Answer: $P = 3{\cdot}14\,t$; $l = 20{\cdot}013$ cm.

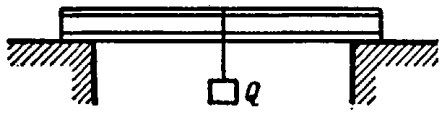


PROBLEM 1.106

1.107. A steel No. 16 I-beam is simply supported, and subjected to a load Q, of 10 t at its mid-span (see figure). The sections of the beam on the supports are prevented from moving horizontally by

friction forces, which are assumed to be applied along the axis of the beam. The coefficient of friction f, between the beam and the supports is 0·3. Determine the maximum temperature change, for which an increase in stresses in the beam is possible, and the magnitude of the maximum temperature stresses.

Answer: $\Delta t = \pm 2{\cdot}3°\text{C}$; $\sigma = \pm 57{\cdot}5\ \text{kg/cm}^2$.



PROBLEM 1.107

1.108. Rails are laid at a temperature t_1 of $+10°\text{C}$ with a gap a of 4 mm. The length of the rails l is 12·5 m. What will be the gap at a temperature t_2 of $-30°\text{C}$? At what temperature will the gap disappear? What will be the stresses at a temperature t_3 of $+50°\text{C}$?

Solution. The contraction of the rail due to the fall in the temperature will be determined by the formula

$$\Delta l = \alpha l \Delta t = 125 \times 10^{-7} \times 1250 \times 40 = 0{\cdot}625\ \text{cm}.$$

Thus

$$a_1 = a + \Delta l = 4 + 6{\cdot}25 = 10{\cdot}25\ \text{mm}.$$

To determine the temperature at which the gap will disappear we shall equate the elongation of the rail with the length of the gap.

$$\alpha l \Delta t = a.$$

From this equation the required temperature difference is

$$\Delta t = \frac{a}{\alpha l} = \frac{0{\cdot}4}{125 \times 10^{-7} \times 1250} = 25{\cdot}6°\text{C}.$$

Consequently, the gap will disappear at the temperature

$$t_4 = t_1 + \Delta t = 10 + 25{\cdot}6 = 35{\cdot}6°\text{C},$$

and the stresses will begin to appear only at $t > t_4$.

They will be

$$\sigma = E\alpha\,(t_4 - t_3) = 2 \times 10^6 \times 125 \times 10^{-7}\,(35{\cdot}6 - 50) = \\ = -360\ \text{kg/cm}^2.$$

1.109. Determine the required gap in rail joints, so that in summer there will be no compression of rails. The laying of the rails is carried out at a temperature of $+10°C$; the maximum summer temperature is $+60°C$; the length of the rails is 8 m. What will be the stresses, if no gaps are left at all?

Answer: 5 mm; $\sigma = -1250\ \text{kg/cm}^2$.

1.110. A steel steam pipe has a length of 30 m at a temperature of $+15°C$. Steam is passed through this pipe, heating it up to $+180°C$. What will then be the length of the pipe, if there is no restraint for its expansion? What will be the stresses in the pipe, if it cannot expand?

Answer: 30·062 m; $\sigma = -4125\ \text{kg/cm}^2$.

1.111. Determine the increase in the stress in a steel tyre, if an iron interlay 0·45 mm thick is inserted between the tyre and the wheel-spider. The diameter of the wheel-spider and the internal diameter of the tyre are both 900 mm. The thickness of the tyre is 45 mm. Neglect the deformation of the interlay and wheel-spider.

Solution. In view of the small thickness of the tyre, in comparison with its diameter, we shall take the mean diameter of the tyre as equal to its internal diameter. The presence of the interlay increases the diameter by $2 \times 0{\cdot}45 = 0{\cdot}9$ mm. The diametral strain of the tyre

$$\varepsilon = \frac{0{\cdot}9}{900} = 0{\cdot}001.$$

The increase in the stresses will be determined by Hooke's law

$$\sigma = E\varepsilon = 2 \times 10^6 \times 0{\cdot}001 = 2000\ \text{kg/cm}^2.$$

1.112. It is proposed to repair a broken cast iron disc by tying it up with a mild steel hoop. The external diameter d of the disc is 80 cm. To fit it the hoop will be heated beforehand. What must be

the internal diameter of the hoop before heating, if the stress σ in it after the fitting is to be 1000 kg/cm²? The disc must be assumed as non-deformable in view of its rigidity in comparison with that of the hoop.

Answer: 79·96 cm.

1.113. A steel band 6 mm thick is to be pulled on to a cylinder 30 cm in diameter. The internal diameter of the hoop formed by the band is 29·98 cm. By how much will it be necessary to increase the temperature of the hoop, so that it would slide over the surface of the cylinder? What stress will occur in the hoop when it reaches the same temperature as the cylinder, if the deformation of the latter may be neglected?

Solution. The mean diameter of the band before heating

$$d = 29{\cdot}98 + 0{\cdot}6 = 30{\cdot}58 \text{ cm}.$$

For the band to slide over the cylinder it is necessary that its internal diameter will become (at least) equal to the diameter of the cylinder, i.e. that it will increase by

$$\Delta d = 30 - 29{\cdot}98 = 0{\cdot}02 \text{ cm}.$$

The mean diameter of the band will increase also by the same amount, Δd. Then the diametral strain

$$\varepsilon = \frac{\Delta d}{d} = \frac{0{\cdot}02}{30{\cdot}58} = 6{\cdot}54 \times 10^{-4}.$$

We shall equate this value of ε with the diametral strain, due to the increase in the temperature:

$$\varepsilon = \alpha \Delta t,$$

whence

$$\Delta t = \frac{\varepsilon}{\alpha} = \frac{6{\cdot}54 \times 10^{-4}}{124 \times 10^{-7}} = 52{\cdot}2°\text{C}.$$

After the band has been put on, it will gradually cool down to the initial temperature, i.e. in comparison with the heated state its temperature will be reduced by an amount Δt of 52·2°C. Since

the cylinder is assumed to be non-deformable the band cannot contract; consequently, to determine the stresses in it we can use the formulae:

$$\sigma = E\alpha\Delta t = 2 \times 10^6 \times 125 \times 10^{-7} \times 52{\cdot}2 = 1308 \text{ kg/cm}^2,$$

or

$$\sigma = E\varepsilon = 2 \times 10^6 \times 6{\cdot}54 \times 10^{-4} = 1308 \text{ kg/cm}^2.$$

1.114. A thin copper ring having an internal diameter of 10 cm is fitted at room temperature, closely but without tension, on to a steel ring having the same thickness. The axial lengths of the rings are equal. If the temperature of both rings is reduced by 50°C, what mean stress is set up in each of them, and by what amount is the internal diameter of the copper ring reduced?

Answer: $\sigma_c = -\sigma_s = 133 \text{ kg/cm}^2$; $\Delta d = -0{\cdot}069$ cm.

1.115. A cylindrical steel ring with a width of 25 cm has an internal diameter of 500 mm and a thickness of 12 mm. Another steel ring of the same width but with a thickness of 18 mm is heated and fitted on to the first one. What must be the internal diameter of the second ring, if after cooling down the stresses in it are not to exceed 700 kg/cm²? Determine also the new internal diameter of the first ring after the second ring has cooled down.

Answer: $d_2 = 523{\cdot}541$ mm; $d_1 = 499{\cdot}731$ mm.

1.116. A steel hoop with an outside diameter of 1000 mm has a thickness of 25 mm. Another hoop of the same material has an inside diameter of 999·5 mm, a thickness of 20 mm, and the same width as the first hoop. The second hoop is heated and placed on the first one. Determine the approximate diameter of the interfaces, the approximate inside diameter of the inner hoop, the compressive stress in the inner hoop and the tensile stress in the outer hoop.

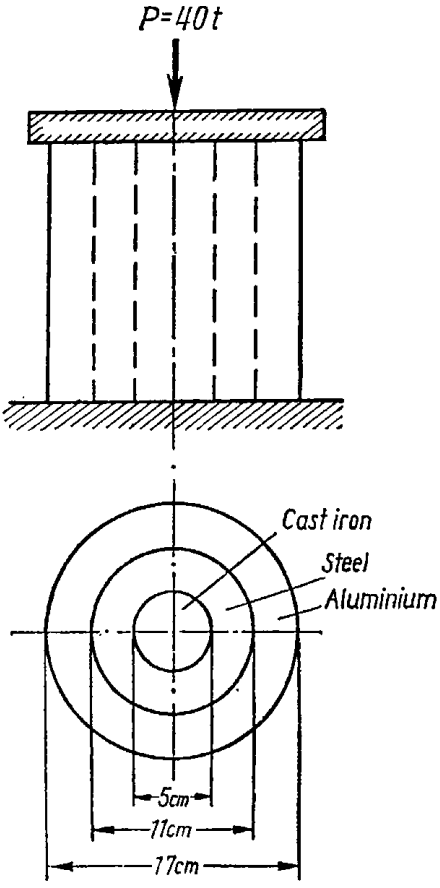
Answer: $d_{\text{inter}} = 999{\cdot}778$ mm; $d_i = 949{\cdot}778$ mm;
$\sigma_i = -444 \text{ kg/cm}^2$; $\sigma_o = 556 \text{ kg/cm}^2$.

1.117. A short cast iron bar is fitted freely into a steel tube which in turn is fitted freely into an aluminium tube (see figure). Determine the stresses in the bar and tubes (neglecting the transverse

deformation) under the influence of the force P compressing the system through a rigid slab.

Solution. The load P will be distributed between the cast iron bar and the steel and aluminium tubes. Equating to zero the sum of all vertical forces, we have

$$P_{c.i.} + P_s + P_a = P.$$



PROBLEM 1.117

This is the only equation of equilibrium for the given arrangement, and we shall have to consider the deformation. The deformation (shortening) is the same for all the three bars, consequently

$$\Delta l_{c.i.} = \Delta l_s = \Delta l_a.$$

Hence it will be possible to write down the two equations of deformation:

$$\Delta l_{c.i.} = \Delta l_s \quad \text{and} \quad \Delta l_{c.i.} = \Delta l_a.$$

Expressing the deformations in terms of the loads, we have

$$\frac{P_{c.i.}\,l}{E_{c.i.}F_{c.i.}} = \frac{P_s l}{E_s F_s} \quad \text{and} \quad \frac{P_{c.i.}\,l}{E_{c.i.}F_{c.i.}} = \frac{P_a l}{E_a F_a}.$$

In the equations of equilibrium and deformation the forces will be expressed in terms of the stresses, noting that

$$\frac{P_{c.i.}}{F_{c.i.}} = \sigma_{c.i.}, \quad \frac{P_s}{F_s} = \sigma_s \quad \text{and} \quad \frac{P_a}{F_a} = \sigma_a.$$

Length l will be eliminated. Then these equations will take the following form:

$$\sigma_{c.i.}F_{c.i.} + \sigma_s F_s + \sigma_a F_a = P; \quad \frac{\sigma_{c.i.}}{E_{c.i.}} = \frac{\sigma_s}{E_s}; \quad \frac{\sigma_{c.i.}}{E_{c.i.}} = \frac{\sigma_a}{E_a}.$$

Solving these three simultaneous equations we obtain the values of the stresses. They are

$$\sigma_{c.i.} = \frac{P}{F_{c.i.}\left(1 + \dfrac{E_s F_s}{E_{c.i.}F_{c.i.}} + \dfrac{E_a F_a}{E_{c.i.}F_{c.i.}}\right)}$$

$$= \frac{40{,}000}{3{\cdot}14 \times 2{\cdot}5^2\left(1 + \dfrac{2}{1{\cdot}2} \cdot \dfrac{5{\cdot}5^2 - 2{\cdot}5^2}{2{\cdot}5^2} \cdot \dfrac{0{\cdot}7}{1{\cdot}2} \times \dfrac{8{\cdot}5^2 - 5{\cdot}5^2}{2{\cdot}5^2}\right)}$$

$$= 180 \text{ kg/cm}^2;$$

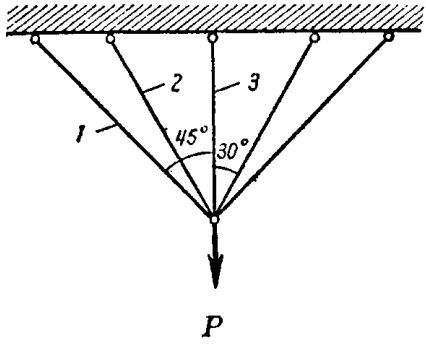
$$\sigma_s = \frac{E_s}{E_{c.i.}}\sigma_{c.i.} = \frac{2 \times 10^6}{1{\cdot}2 \times 10^6} \times 180 = 300 \text{ kg/cm}^2;$$

$$\sigma_a = \frac{E_a}{E_{c.i.}}\sigma_{c.i.} = \frac{0{\cdot}7 \times 10^6}{1{\cdot}2 \times 10^6} \times 180 = 105 \text{ kg/cm}^2.$$

All the stresses are compressive.

1.118. All rods supporting a load P of 30 t (see figure) are of identical circular cross-section 3 cm in diameter, and are made of the same material. Determine the stresses in the rods.

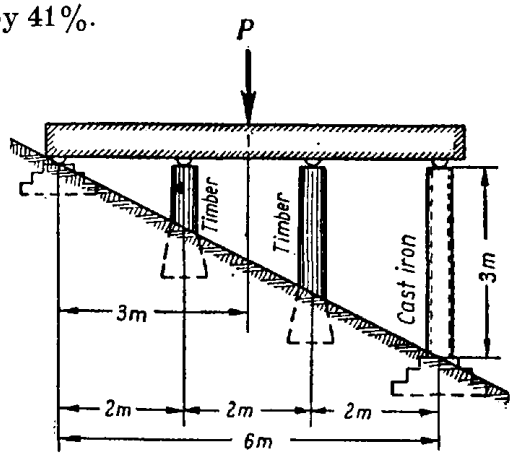
Answer: $\sigma_1 = 707$ kg/cm^2; $\sigma_2 = 1060$ kg/cm^2; $\sigma_3 = 1414$ kg/cm^2.



PROBLEM 1.118

1.119. A rigid beam was originally supported at its right-hand end on a cast iron stanchion having a cross-sectional area of 100 cm^2, and did not have any intermediate supports. Then two intermediate wooden stanchions of circular cross-section 25 cm in diameter were provided (see figure). By what percentage can the load be increased after the installation of the intermediate stanchions, with the condition that the stresses in the cast iron stanchion will remain as they were originally?

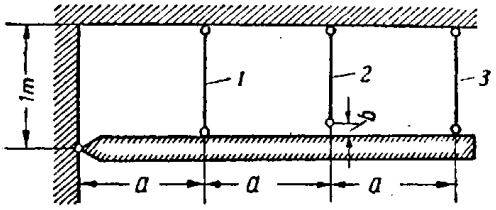
Answer: By 41%.



PROBLEM 1.119

1.120. A rigid bar is suspended from three steel rods of identical cross-section with an area of 20 cm² each. The middle rod is made 0·5 mm (= δ) shorter than its design length (see figure). Determine the stresses in the rods after the assembly of the structure.

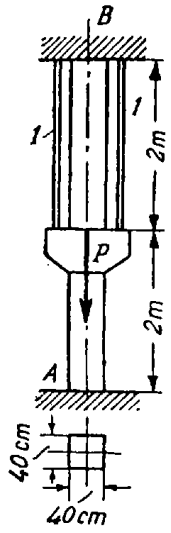
Answer: $\sigma_1 = -143 \text{kg/cm}^2$; $\sigma_2 = 715 \text{kg/cm}^2$; $\sigma_3 = -429 \text{kg/cm}^2$.



PROBLEM 1.120

1.121. A concrete column AB clamped at both ends (see figure) is loaded axially in its middle section by a force P of 32 t. What cross-sectional area must the two steel ties 1 have if the tensile stress in the upper part of the concrete column is not to exceed 5 kg/cm²? What are then the stresses in the ties and in the lower part of the column? The modulus of elasticity (E) for the concrete is to be taken the same in tension and compression, namely $1{\cdot}5 \times 10^5$ kg/cm².

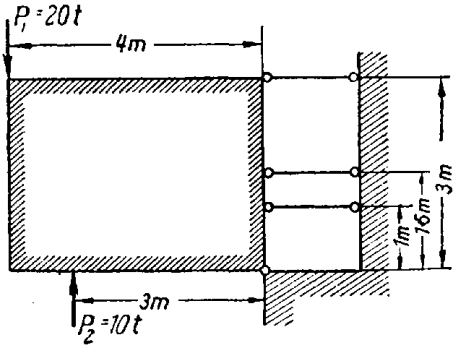
Answer: $F = 2 \times 120$ cm²; $\sigma_{\text{ties}} = 66{\cdot}7$ kg/cm²; $\sigma_c = -5$ kg/cm².



PROBLEM 1.121

1.122. A rigid structure (see figure) is fastened to the foundation by means of a hinge and three steel ties with identical cross-sections and lengths. Choose a cross-section for the ties, if the allowable stress $[\sigma]$ is 1600 kg/cm^2.

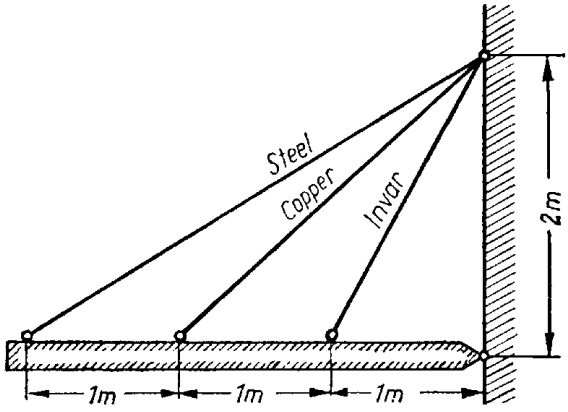
Answer: 7·65 cm^2.



PROBLEM 1.122

1.123. A rigid bar is suspended from three rods of identical cross-section. They each have an area of 10 cm^2, but are made of different materials (see figure). Determine the stresses in the rods after increasing their temperature by 30°C ($= \Delta t$). For invar $\alpha_1 = 0$ and $E_1 = 2 \times 10^6$ kg/cm^2.

Answer: $\sigma_s = -142$ kg/cm^2; $\sigma_c = -166$ kg/cm^2; $\sigma_1 = 527$ kg/cm^2.



PROBLEM 1.123

3. SELF-WEIGHT

1.124. Determine the compressive stress in the base of a stonework pillar of constant cross-section having a height of 8 m. The weight of 1 m^3 of masonry is 2 t.

Answer: 1·6 kg/cm^2.

1.125. A lift cage weighing 1·6 t is suspended from two cables at a depth h (= 140 m). The cross-sectional area of each cable is 1·25 cm^2; it weighs 1·5 kg/m. What are the stresses in the upper and lower sections of the cable?

Answer: $\sigma_l = 640$ kg/cm^2; $\sigma_u = 808$ kg/cm^2.

1.126. Determine the volume of a stonework support, compressed by a force P of 400 t, in the three cases which follow: (1) the support is of constant cross-section, (2) the support is stepped, in three parts of equal length, and (3) the support is made in the form of a block offering constant resistance to compression. Take the allowable stress as 12 kg/cm^2, the specific weight of the stonework as 2·2 t/m^3, and the height of the support as 42 m.

Answer: $V_1 = 610$ m^3; $V_2 = 261$ m^3; $V_3 = 209$ m^3.

1.127. Determine the dimensions of a square cross-section of a stonework pillar 10 m high, centrally loaded by a force of 50 t. The allowable compressive stress is 10 kg/cm^2, the specific gravity of the stonework is 2. Draw a diagram showing the stress variation along the length of the pillar.

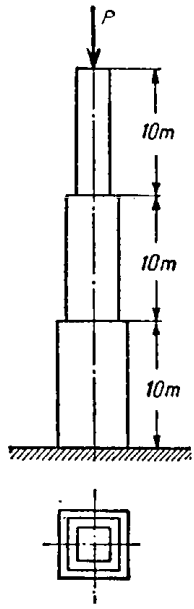
Answer: 79 × 79 cm^2.

1.128. Determine the required diameter of a mine shaft rod 100 m long, fixed at its upper end, and loaded at its lower end by a force of 2 t. The allowable stress $[\sigma]$ for the material of the rod is 1000 kg/cm^2. Carry out the calculation (i) with the weight of the rod taken into account, and (ii) without the weight of the rod taken into account.

Answer: (i) 1·66 cm; (ii) 1·59 cm.

1.129. Determine the dimensions in plan of a stepped pillar of square cross-section 30 m high, having three portions of equal length, and being compressed by a force P of 60 t (see figure). The allowable stress $[\sigma]$ for the masonry of the pillar in compression is 10 kg/cm^2; its specific gravity is 2. Construct a diagram showing the stress variation along the length of the pillar.

Answer: 87 cm × 87 cm; 97 cm × 97 cm; 108 cm × 108 cm.

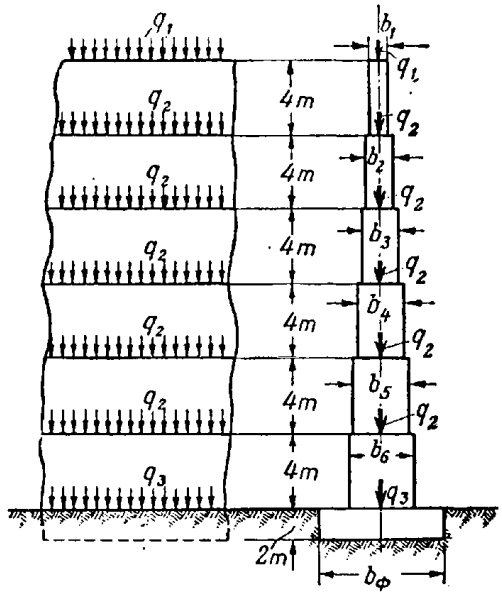


PROBLEM 1.129

1.130. In the figure a part of the façade and a transverse section through the brick wall of a six-storey building is represented. The following loads act on the wall per linear metre: from the roofing a load q_1 of 6 t/m, from each of the intermediate floors a load q_2 of 4 t/m, and from the ground floor a load q_3 of 3 t/m. These loads are applied along the axis of the wall. The specific weight of the material of the wall is 2 t/m^3. Determine the minimum thickness of the walls at each storey if the allowable stress $[\sigma]$ is 6 kg/cm^2, and the width of the foundation if the allowable ground pressure is 5 kg/cm^2.

Answer: $b_1 = 11{\cdot}5$ cm; $b_2 = 21$ cm; $b_3 = 32$ cm; $b_4 = 44{\cdot}5$ cm; $b_5 = 59$ cm; $b_6 = 76$ cm; $b_7 = 125$ cm.

Due to thermal considerations these minimum dimensions are substantially increased.



PROBLEM 1.130

1.131. A steel rod is suspended vertically from its upper end and loaded only by its own weight. What is the maximum allowable length of the rod, if the stress is not to exceed 300 kg/cm^2, and the specific gravity of the steel is 7·85? Draw a diagram showing the stress distribution along the length of the rod.

Answer: 382 m.

1.132. For a steel rod, 120 m long, determine the total elongation due solely to its own weight.

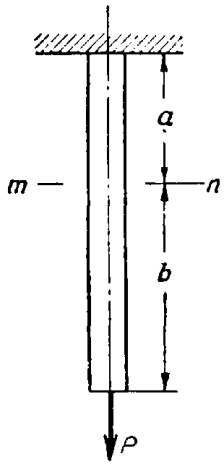
Answer: 2·8 mm.

1.133. A vertical steel rod, with a cross-sectional area F and a length l of 300 m, has been working under a load P, which induced a stress of 6 kg/mm^2 in it. Altering neither the dimensions of the

cross-section of the rod nor the load, it is required to increase the length of the rod, assuming that the allowable stress may be increased to 6·5 kg/mm². How much longer can the rod be made?

Answer: 64·1 m.

1.134. Determine the displacement of section mn of the bar represented in the figure, if its cross-section is F, the modulus of elasticity is E, and the specific weight of the material is γ. Take the weight of the bar into account.



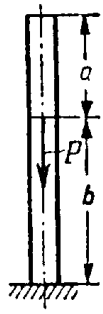
PROBLEM 1.134

Solution. In relation to the upper part (of length a) the weight of the lower part of the bar (of length b) will act together with force P as an external load. Thus the displacement of the section mn, equal to the elongation of the part of length a of the bar, will be

$$\delta_{mn} = \Delta l_a = \frac{(P + \gamma F b)\, a}{EF} + \frac{\gamma a^2}{2E}.$$

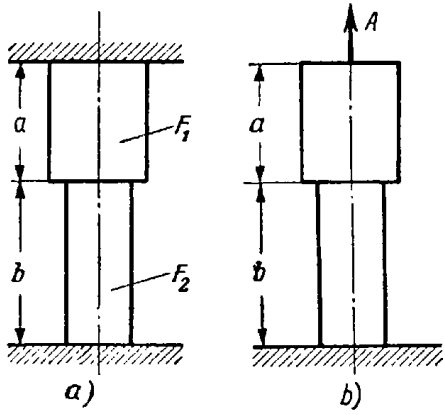
1.135. Determine, by taking the weight of the bar into account, the displacement of the free end of the bar represented in the figure, if its cross-section is F, the modulus of elasticity is E, and the specific weight of the material is γ.

Answer: $\dfrac{Pb}{EF} + \dfrac{\gamma(b+a)^2}{2E}$.



PROBLEM 1.135

1.136. Determine the reactions at the supports of the bar represented in figure (a), considering only the weight of the bar. The upper and lower parts of the bar are made of the same material with a specific weight γ and modulus E, but the cross-sections are different.



PROBLEM 1.136

Solution. We shall denote the upper reaction by A and the lower reaction by B, and assume that both of them are directed upwards. Then the only equation of equilibrium, the sum of the vertical forces, will state that the weight of the rod is equal to the sum of the reactions:

$$\gamma F_1 a + \gamma F_2 b = A + B.$$

Thus, the problem is statically indeterminate. We shall consider the deformations. To obtain a simply supported system we discard the upper support. We shall load this system by the reaction A (see figure (b)). Since the total deformation is zero the elongation of the bar caused by the reaction A must be equal to its contraction due to the weight. This condition will provide the supplementary equation of deformation. The elongation of the bar due to the reaction A

$$\Delta l_A = \frac{A}{E}\left(\frac{b}{F_2} + \frac{a}{F_1}\right) = \frac{Aa}{EF_1}\left(1 + \frac{b}{a} \times \frac{F_1}{F_2}\right).$$

The contraction of the bar caused by its weight

$$\Delta l_Q = \frac{\gamma a^2}{2E} + \frac{\gamma b^2}{2E} + \frac{\gamma a F_1 b}{EF_2} = \frac{\gamma a^2}{2E}\left[1 + \left(\frac{b}{a}\right)^2 + 2\frac{b}{a} \times \frac{F_1}{F_2}\right].$$

Equating the values of these quantities we have

$$\frac{Aa}{EF_1}\left(1 + \frac{b}{a} \times \frac{F_1}{F_2}\right) = \frac{\gamma a^2}{2E}\left[1 + \left(\frac{b}{a}\right)^2 + 2\frac{b}{a} \times \frac{F_1}{F_2}\right].$$

Hence

$$A = \frac{\gamma F_1 a}{2} \times \frac{1 + \frac{b^2}{a^2} + 2\frac{b}{a} \times \frac{F_1}{F_2}}{1 + \frac{b}{a} \cdot \frac{F_1}{F_2}},$$

and from the equation of equilibrium

$$B = \gamma F_1 a + \gamma F_2 b - A.$$

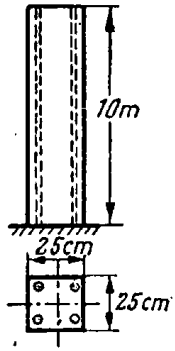
1.137. A bar of length l and cross-sectional area F, made of a material with a specific weight γ, is clamped at its upper and lower ends as indicated in the figure. Determine the stresses in the upper and lower sections of the bar, due to its own weight, and draw a diagram showing the stress distribution along the length of the bar.

Answer: $\sigma_u = \frac{\gamma l}{2}$; $\sigma_l = -\frac{\gamma l}{2}$.

PROBLEM 1.137

1.138. A reinforced concrete column with the dimensions indicated in the figure is loaded only by its own weight. Determine the stresses in the concrete and in the reinforcement of the column, if the specific weight of the reinforced concrete is 2·5 t/m³. The ratio of the moduli of elasticity of the steel reinforcement and the concrete is to be taken as 15, and the diameter of reinforcing rods as 25 mm.

Answer: $\sigma_c = -1{\cdot}7$ kg/cm²; $\sigma_s = -25{\cdot}5$ kg/cm².



PROBLEM 1.138

1.139. A concrete column of constant cross-section is rigidly clamped at its upper and lower ends. What is the limiting length l

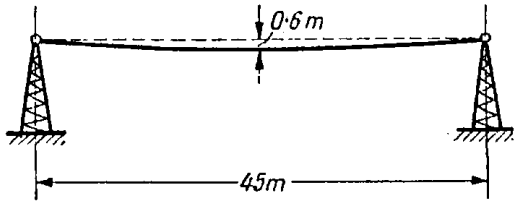
of the column, if the tensile stress in the concrete is not to exceed 5 kg/cm²? The specific weight of the concrete is to be taken to equal 2·5 t/m³.

Answer: 40 m.

4. CALCULATION OF FLEXIBLE WIRES AND CABLES

1.140. A cable 5 cm in diameter and weighing 1 kg/m is suspended (see figure), with a sag of 0·5 m, from two supports located at the same level 45 m apart. Determine the maximum and minimum tensile forces in the cable.

Answer: $H = 422$ kg; $T_{\max} \approx H$.



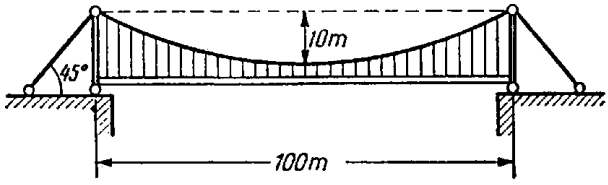
PROBLEM 1.140

1.141. A suspension bridge is suspended from two chains. The span of the chains is 100 m, and the sag is 8 m. The load of the bridge is 6 t/m. Determine the required cross-section of the chains (assuming that they are of constant cross-section over the whole length), if $[\sigma]$ is 1400 kg/cm².

Answer: 352 cm².

1.142. The chain of the suspension bridge represented in the figure is loaded by a uniformly distributed load with an intensity q of 5 t/m. Determine the necessary cross-section of the stay ropes, if the allowable stress $[\sigma]$ for them is 1000 kg/cm².

Answer: 884 cm².



PROBLEM 1.142

1.143. A copper wire with a cross-sectional area of 95 mm² is suspended from two supports located, at the same level, a distance l of 100 m from each other. The sag f of the wire is 4·0 m. Find the increase of the stresses on reducing the temperature by 25°C, if the weight q of the wire is 0·862 kg/m, the coefficient of linear expansion ($= \alpha$) is 167×10^{-7}/°C, and the modulus of elasticity E is 12,000 kg/mm².

Answer: The stresses are increased by 14·7 kg/cm².

1.144. A steel wire is to be suspended from two points located at the same level 30 m apart. The wire is hung at a temperature of + 35°C. What must be the initial sag for the stresses not to exceed 1200 kg/cm² at a temperature of − 25°C? What will be the required length of the wire?

Solution. The condition of the limiting stress of the wire is expressed as

$$\sigma = \frac{H}{F} = \frac{ql^2}{8fF} \leq [\sigma].$$

Since the only load on the wire will be its own weight $q = \gamma \times F \times 1$, the condition of the limiting stress is

$$\frac{\gamma F l^2}{8fF} \leq [\sigma].$$

After eliminating the cross-sectional area of the wire we find the required sag

$$f = \frac{\gamma l^2}{8\,[\sigma]} = \frac{7{\cdot}8 \times 10^{-3} \times 30^2 \times 10^4}{8 \times 1200} = 7{\cdot}31 \text{ cm}.$$

The wire must have this sag while in the critical state, i.e. at the temperature of −25°C. In order to find the sag when the wire was hung, it is necessary to connect its state at $t_1 = -25$°C with its state at $t_2 = +35$°C. It is known that the two states of the wire are connected by the following cubic equation:

$$H_2^3 + \left[\frac{EFq_1^2l^2}{24H_1^2} + EF\alpha\,(t_2 - t_1) - H_1\right]H_2^2 - \frac{EFq_2^2l^2}{24} = 0.$$

Making the following substitutions

$$H_2 = \frac{ql^2}{8f_2}, \quad H_1 = H = [\sigma]F \quad \text{and} \quad q_1 = q_2 = q = \gamma F,$$

the equation is written as:

$$f_2^3 - \frac{3}{8} l^2 \left[\frac{1}{24} \times \frac{\gamma^2 l^2}{[\sigma]^2} + \alpha(t_2 - t_1) - \frac{[\sigma]}{E} \right] f_2 - \frac{3\gamma l^4}{64E} = 0.$$

After substituting the numerical data we have

$$f_2^3 - \frac{3}{8} \times 3000 \left[\frac{0{\cdot}0078^2 \times 3000^2}{24 \times 1200^2} + \right.$$

$$\left. + 125 \times 10^{-7}(35 + 25) \frac{1200}{2 \times 10^6} \right] f_2 -$$

$$- \frac{3 \times 0{\cdot}0078 \times 3000^4}{64 \times 2 \times 10^6} = 0,$$

which, after cancelling out, reduces to

$$f_2^3 - 560 f_2 - 14{,}850 = 0.$$

Hence $f = 32$ cm. This will be the initial sag. The length of the wire having this sag will be found from the formula

$$s = l\left(1 + \frac{8}{3} \times \frac{f_2^2}{l^2}\right) = 30\left(1 + \frac{8}{3} \times \frac{0{\cdot}32^2}{30^2}\right) = 30{\cdot}009 \text{ m}.$$

1.145. A multi-strand copper wire is suspended, at a temperature t_1 of $+15°C$, between two supports located at the same level 10 m (= distance l) from each other. The cross-sectional area F of the wire is 120 mm^2. Later the temperature falls to $t_2 = -5°C$, and the wire is covered by a layer of ice 10 mm thick. Simultaneously the wire is subjected to a horizontal wind pressure with an intensity of 24 kg/m^2 of the vertical projection of the wire covered by the layer of ice. Further, it is envisaged that the temperature may fall to $t_3 = -40°C$, but without simultaneous icing and wind. What must be the initial sag of the wire in order that under the

worst conditions the stresses in it would not exceed 800 kg/cm², if the weight of 1 linear metre of the wire is 1·09 kg. Its outside diameter is 14·2 mm; $\alpha = 17 \times 10^{-6}/°C$; $E = 1{\cdot}3 \times 10^4$ kg/mm². The specific gravity of the ice is 0·9.

Answer: $f = 2{\cdot}59$ m.

1.146. Determine the tension in the steel and aluminium parts of a steel–aluminium cable suspended from two supports at the same level, having a span l and a sag f. The cross-sectional area of the steel in the cable is F_s, that of the aluminium is F_a. The specific weights and the moduli of elasticity are γ_s, γ_a, E_s, and E_a respectively.

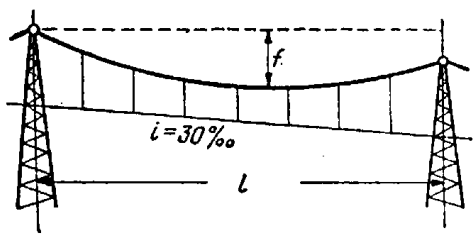
Answer: $$H_s = \frac{(\gamma_s F_s + \gamma_a F_a) l^2}{8f\left(1 + \frac{E_a F_a}{E_s F_s}\right)}; \qquad H_a = \frac{(\gamma_s F_s + \gamma_a F_a) l^2}{8f\left(1 + \frac{E_s F_s}{E_a F_a}\right)}.$$

1.147. The horizontal distance between the supports of a wire, with a cross-sectional area of 1 cm and a weight of 8 g/cm³, is 37 m. One support is 30 cm lower than the other, and the lowest point of the wire is located 90 cm below the level of the lower support. Determine the horizontal distance from the lower support to the lowest point of the wire, and the tension at that point.

Answer: 17·1 m; $H = 131$ kg.

1.148. Determine the tension in the lowest section of a suspension cable carrying the conductor of an electric railway (see figure). The incline i, is $30^0/_{00}$ the span l is 75 m, and the load q is 1·6 kg/m. The distance f, along the vertical, to the lowest point of the cable from the top of the higher support is 3 m.

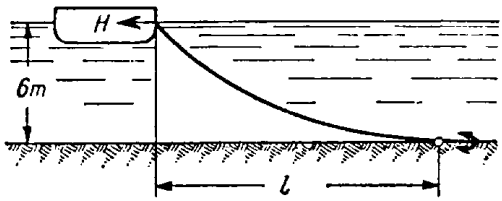
Answer: $H = 667$ kg.



PROBLEM 1.148

1.149. The anchor chain holding a barge is of steel, with a cross-sectional area of 5 cm². At the point where it is attached to the anchor it touches the bottom horizontally (see figure). Determine the distance l from the barge to the anchor, if the magnitude of the horizontal force pulling the anchor is 100 kg.

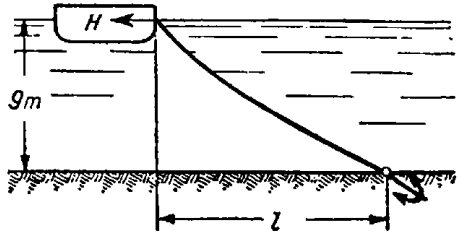
Answer: 18·8 m.



PROBLEM 1.149

1.150. A barge transmits a horizontal force H of 36 t to the anchor chain (see figure). The total length of the chain is 45 m, and the vertical distance (from the point where the chain is attached to the barge) to the bottom is 9 m. The weight of the chain in water is 50 kg/m. Determine the horizontal distance between the anchor and the point where the chain is suspended from the barge.

Answer: 39·66 m.

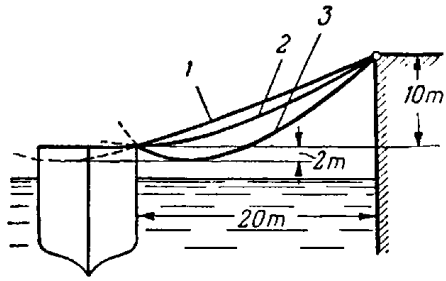


PROBLEM 1.150

1.151. A ship is moored to the bank by a steel rope with a weight of 3 kg/m and a cross-sectional area of 4 cm² (see figure). With what force does the ship pull away from the bank, if the rope sags along the curves 1, 2, and 3? The lowest point of curve 1 is on the left of its left-hand fixing, the lowest point of curve 2 is at its left-hand

fixing, and the lowest point of curve 3 is on the right of its fixing—and at the same level as the lowest point of curve 1.

Answer: $H_1 = 143$ kg; $H_2 = 60$ kg; $H_3 = 25$ kg.



PROBLEM 1.151

CHAPTER 2

STATE OF COMPOUND STRESS*

5. UNIAXIAL AND PLANE STATES OF STRESS

2.1. A rod 6 cm in diameter is pulled by a force of 25 t. Determine the normal and shear stresses in a plane the normal of which makes an angle of 30° with the axis of the rod. Determine in which plane the shear stresses will attain a maximum, and calculate their magnitude.

Solution. We shall find first the stress in a plane perpendicular to the axis of the rod:

$$\sigma_0 = \frac{P}{F} = \frac{P}{\frac{\pi d^2}{4}} = \frac{4 \times 25{,}000}{3{\cdot}14 \times 6^2} = 884 \text{ kg/cm}^2.$$

The normal and shear stresses in an inclined plane are determined by the formulae:

$$\sigma_\alpha = \sigma_0 \cos^2 \alpha = 884 \times 0{\cdot}866^2 = 663 \text{ kg/cm}^2,$$

$$\tau_\alpha = \frac{\sigma_0}{2} \sin 2\alpha = \frac{884}{2} \times 0{\cdot}866 = 383 \text{ kg/cm}^2.$$

The shear stresses attain a maximum in planes inclined at an angle of 45° to the axis of the rod. Their magnitude is

$$\tau_{\max} = \frac{\sigma_0}{2} = \frac{884}{2} = 442 \text{ kg/cm}^2.$$

* For notes on Theories of Strength, see Appendix B, Section 2.

2.2. A rod 7·5 cm in diameter is pulled by a force of 35 t. Determine the resultant stress in a transverse plane. Determine also the total and shear stresses in a plane the normal of which is inclined at an angle of 15° to the axis of the rod.

Answer: $\sigma_0 = 795$ kg/cm^2; $p_\alpha = 767$ kg/cm^2; $\tau_\alpha = 198$ kg/cm^2.

2.3. A rod of circular cross-section is pulled by a force of 15 t. The shear stress in any plane must not exceed 600 kg/cm^2. Determine the diameter of the rod.

Answer: 4 cm.

2.4. In a rod in tension the normal stresses in one inclined plane are 700 kg/cm^2, and the shear stresses are 500 kg/cm^2. Determine the maximum normal and shear stresses.

Answer: $\sigma_{\max} = 1056$ kg/cm^2; $\tau_{\max} = 528$ kg/cm^2.

2.5. The normal stress in a transverse plane of a rod in tension is 500 kg/cm^2. Thereby in an inclined plane the shear stress is 160 kg/cm^2. Determine the angular position of this plane, and the magnitude of the normal stress acting in it.

Answer: $\alpha_1 = 19°55'$; $\sigma_{\alpha 1} = 442$ kg/cm^2;
$\alpha_2 = 70°\ 5'$; $\sigma_{\alpha 2} = 58$ kg/cm^2.

2.6. A short concrete bar 20×20 cm^2 in cross-section is compressed by a force P. What is the force P, if the normal compressive stress in a plane inclined at 45° to the axis of the rod is 15 kg/cm^2?

Answer: 12 t.

2.7. The resultant stress in a plane passing through a given point in a structural element is 300 kg/cm^2. This stress is inclined at an angle of 60° to the plane. In another plane perpendicular to the first only shear stresses act. Find the maximum tensile stress at this point.

Answer: 328 kg/cm^2.

2.8. Given σ_1 and σ_2 (figure (a)), or σ_1 and σ_3 (figure (b)) determine graphically σ_α, τ_α and the resultant stress p_α in a plane at an angle α to the plane in which σ_1 acts. In addition, determine the maximum shear stress in the material.

(a) $\sigma_1 = 800\ \text{kg/cm}^2$, $\sigma_3 = -500\ \text{kg/cm}^2$, $\alpha = 60°$;

(b) $\sigma_1 = 750\ \text{kg/cm}^2$, $\sigma_3 = -750\ \text{kg/cm}^2$, $\alpha = 60°$;

(c) $\sigma_1 = 150\ \text{kg/cm}^2$, $\sigma_2 = 150\ \text{kg/cm}^2$, $\alpha = 60°$;

(d) $\sigma_1 = 150\ \text{kg/cm}^2$, $\sigma_2 = 150\ \text{kg/cm}^2$, $\alpha =$ any angle;

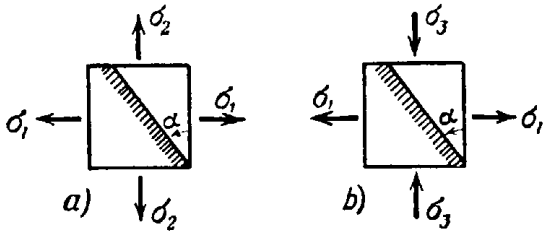
(e) $\sigma_1 = 150\ \text{kg/cm}^2$, $\sigma_3 = -150\ \text{kg/cm}^2$, $\alpha = 45°$;

(f) $\sigma_1 = 1000\ \text{kg/cm}^2$, $\sigma_3 = -600\ \text{kg/cm}^2$, $\alpha = 60°$;

(g) $\sigma_1 = 140\ \text{kg/cm}^2$, $\sigma_3 = -140\ \text{kg/cm}^2$, $\alpha = 30°$;

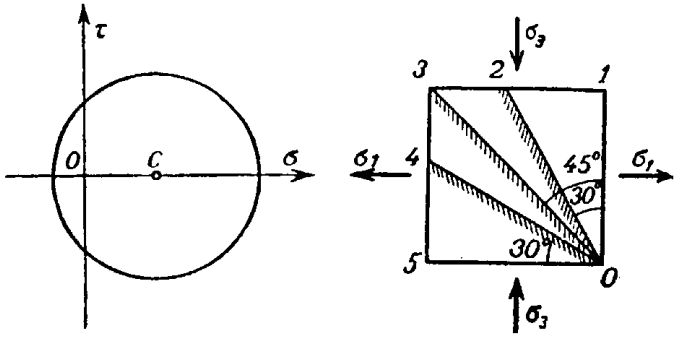
(h) $\sigma_1 = 600\ \text{kg/cm}^2$, $\sigma_3 = -300\ \text{kg/cm}^2$, $\alpha = 60°$.

Answer: (a) $-175\ \text{kg/cm}^2$; (b) $-375\ \text{kg/cm}^2$; (c) $150\ \text{kg/cm}^2$; (d) $150\ \text{kg/cm}^2$; (e) 0; (f) $-200\ \text{kg/cm}^2$; (g) $70\ \text{kg/cm}^2$; (h) $-75\ \text{kg/cm}^2$. The values given are of σ_α.



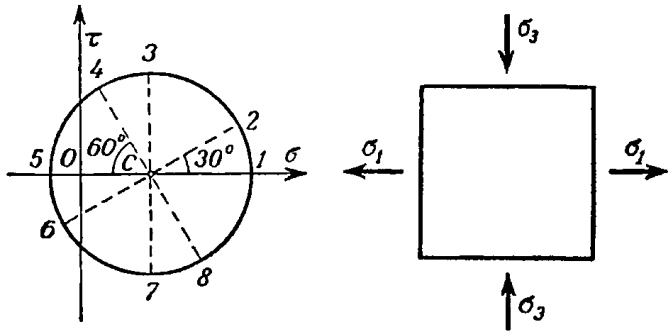
PROBLEM 2.8

2.9. Mohr's circle is constructed for an element in a plane state of stress (see figure). Indicate on that circle the points defining the stresses in planes 1–0, 2–0, 3–0 and 5–0.



PROBLEM 2.9

2.10. Mohr's circle is constructed for an element in a plane state of stress (see figure). Indicate on the element the planes on which the stresses are determined by the co-ordinates of points 1, 2, 3, 4, 5, 6, 7, and 8 on the circle.



PROBLEM 2.10

2.11. A square steel plate with sides of 15 cm is pulled in one direction by a stress of 750 kg/cm², whereas in the other direction it is compressed by a stress equal in magnitude. Find the strain on the diagonals of the plate.

Answer: $\varepsilon = 0$.

2.12. A steel sheet is elongated 0·2 mm over a length of 250 mm, with a tensile stress of 1500 kg/cm². If an additional stress of 1500 kg/cm² acts in the sheet in the direction perpendicular to the first, what will then be the elongation over a length of 250 mm in the two perpendicular directions? Assume that Poisson's ratio μ is 0·3. How will the deformation be altered, if the additional stress is in the same direction but compressive?

Answer: With the additional stress tensile: $\Delta l_{\parallel} = \Delta l_{\perp} = 0{\cdot}14$ mm; with the additional stress compressive: $\Delta l_{\parallel} = 0{\cdot}26$ mm and $\Delta l_{\perp} = -0{\cdot}26$ mm.

2.13. The principal stresses in the wall of a container are 150 kg/cm² and 300 kg/cm². Determine the normal, shear and resultant stresses in magnitude and direction, in a plane the normal of which forms an angle of 30° with the direction of the maximum principal stress.

Find the plane normal to the wall having the maximum shear stress, and calculate both components of the stress in this plane.

Answer: $\sigma_\alpha = 263$ kg/cm^2; $\tau_\alpha = 65$ kg/cm^2; $p_\alpha = 271$ kg/cm^2 at an angle of 16° to the direction of σ_1; the plane with the maximum shear stress lies at an angle of 45° to the plane of maximum principal stress; in this plane: $\sigma = 225$ kg/cm^2 and $\tau_{max} = 75$ kg/cm^2.

2.14. The principal stresses in the wall of a boiler are 600 kg/cm^2 and 300 kg/cm^2. Determine the strains in the principal directions. What stress causes the same maximum strain in a uniaxial state of stress? Calculate the amount of strain energy stored in 1 cm^3 of the wall of the boiler.

Answer: $\varepsilon_1 = 2{\cdot}55 \times 10^{-4}$; $\varepsilon_2 = 0{\cdot}6 \times 10^{-4}$; $\sigma = 510$ kg/cm^2; $a = 8{\cdot}55 \times 10^{-2}$ kg/cm^2.

2.15. The principal stresses at a point in an element subjected to plane stress are 1200 and 450 kg/cm^2. Determine the position of the plane on which the total stress is inclined at a minimum angle.

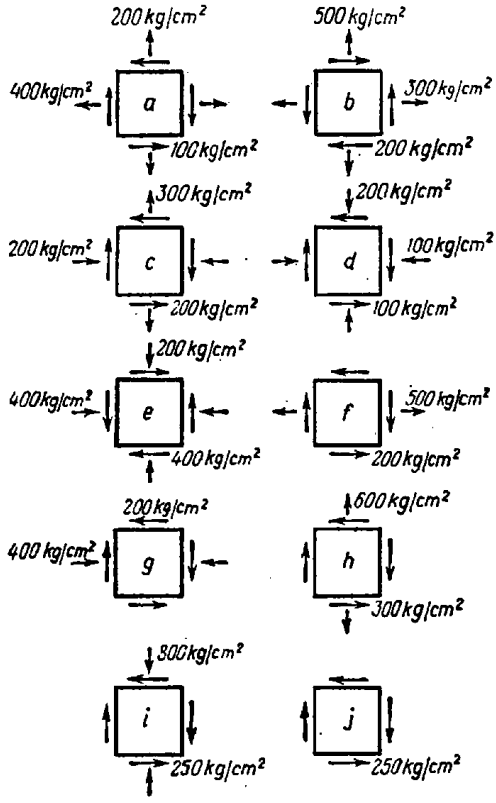
Answer: At an angle of 58°30′ to the plane of maximum principal stress.

2.16. Determine graphically (by means of Mohr's circle) the magnitude and direction of the principal stresses for the elements shown in the figure, and indicate inside them the planes subjected to the principal stresses alone.

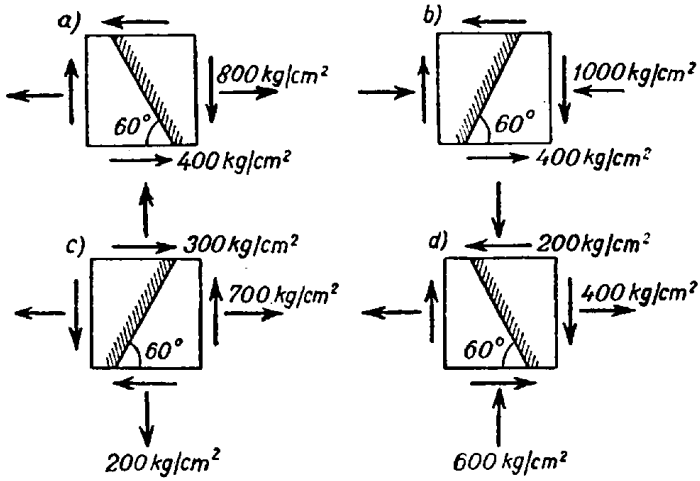
Answer: (a) 440 kg/cm^2; (b) 625 kg/cm^2; (c) 370 kg/cm^2; (d) -40 kg/cm^2; (e) -110 kg/cm^2; (f) 570 kg/cm^2; (g) 83 kg/cm^2; (h) 730 kg/cm^2; (i) 70 kg/cm^2; (j) 250 kg/cm^2. The values given are of σ_1.

2.17. Given σ_x, σ_y, τ_x and τ_y (see figure) determine graphically σ_α and τ_α on the inclined plane indicated.

Answer: (a) $\sigma_\alpha = 258$ kg/cm^2, $\tau_\alpha = 546$ kg/cm^2;
(b) $\sigma_\alpha = -402$ kg/cm^2, $\tau_\alpha = 630$ kg/cm^2;
(c) $\sigma_\alpha = 314$ kg/cm^2, $\tau_\alpha = 368$ kg/cm^2;
(d) $\sigma_\alpha = -24$ kg/cm^2, $\tau_\alpha = 532$ kg/cm^2.



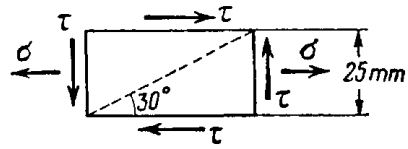
PROBLEM 2.16



PROBLEM 2.17

2.18. In an element (see figure) cut out of a steel component the following stresses act: $\sigma = 300$ kg/cm^2 and $\tau = 150$ kg/cm^2. Determine the change in the angle of inclination, and in the length of the diagonal.

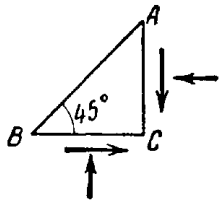
Answer: $\Delta l_{\text{diag.}} = 9{\cdot}25 \times 10^{-3}$ mm.



PROBLEM 2.18

2.19. A prismatic element of a material is shown in the figure. Plane AB and the base of the prism (parallel to ABC) are not loaded; the normal stresses in planes AC and BC are both 150 kg/cm^2. Determine the shear stresses in planes AC and BC, and the magnitude and direction of the principal stresses.

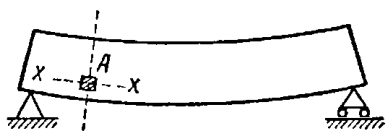
Answer: $\tau = 150$ kg/cm^2; $\sigma_1 = 0$; $\sigma_3 = -300$ kg/cm^2.



PROBLEM 2.19

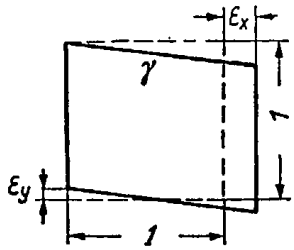
2.20. Under the influence of a moving train the strains in a longitudinal girder of a steel bridge were measured at point A by means of strain gauges (see figure). The strain, ε_x, in the direction of the xx axis (parallel to the axis of the girder) was found to be 0·0004; the strain, ε_y, perpendicular to the former, was $-0{\cdot}00012$. Determine the normal stresses in the directions along and across the axis of the girder.

Answer: Parallel to the axis: $\sigma = 800$ kg/cm^2; perpendicular to the axis: $\sigma = 0$.



PROBLEM 2.20

2.21. The strains of a plane element of a steel component (see figure) are: $\varepsilon_x = 5{\cdot}32 \times 10^{-4}$, $\varepsilon_y = -1{\cdot}82 \times 10^{-4}$ and $\gamma = 1{\cdot}19 \times 10^{-3}$. Determine the magnitude and direction of the principal stresses.



PROBLEM 2.21

Answer: $\sigma_1 = 1600$ kg/cm², $\sigma_3 = -600$ kg/cm²; σ_1 is rotated through 30° clockwise from the horizontal.

6. THREE-DIMENSIONAL STATE OF STRESS

2.22. Determine the maximum strains of an element in a three-dimensional state of stress, if E is 2×10 kg/cm² and μ is 0·25. The principal stresses have the following values:

(a) $+150$ kg/cm², $+300$ kg/cm², $+450$ kg/cm²;

(b) $+150$ kg/cm², $+300$ kg/cm², -450 kg/cm²;

(c) $+150$ kg/cm², -300 kg/cm², -450 kg/cm².

Answer: (a) $\varepsilon_1 = 1685 \times 10^{-7}$; (b) $\varepsilon_2 = -2812 \times 10^{-7}$; (c) $\varepsilon_3 = -2062 \times 10^{-7}$.

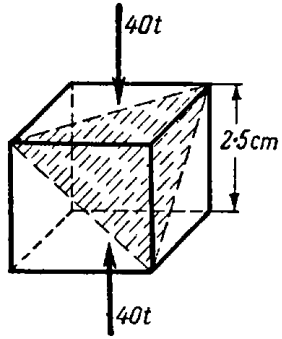
2.23. A steel element in the form of a parallelepiped is subjected to stresses in three mutually perpendicular directions. They are $+750$ kg/cm², -600 kg/cm², $+900$ kg/cm². Determine the strains in all three directions, and the value of the bulk modulus. Calculate the effective stresses using the third and fourth theories of strength.

Answer: $\varepsilon_1 = 4{\cdot}275 \times 10^{-4}$; $\varepsilon_2 = 3{\cdot}3 \times 10^{-4}$; $\varepsilon_3 = -0{\cdot}548 \times 10^{-3}$; $K = 1{\cdot}667 \times 10^6$ kg/cm²; $\sigma_{r3} = 1500$ kg/cm²; $\sigma_{r4} = 1430$ kg/cm².

2.24. A cube with an edge of 2·5 cm (see figure) was tested in compression. Under a load of 40 t it failed along a plane passing

through the diagonal of the top, and through the diagonals of the adjacent sides. Determine the resultant, normal and shear stresses in this plane at the instant of the failure.

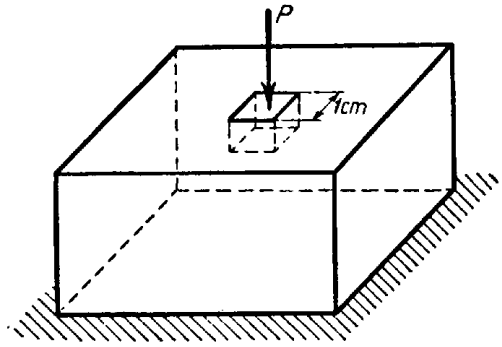
Answer: $p_\alpha = -3680$ kg/cm^2; $\sigma_\alpha = -2133$ kg/cm^2; $\tau_\alpha = 3010$ kg/cm^2.



PROBLEM 2.24

2.25. A recess, in the form of a cube of 1 cm^3, is made in a thick steel slab (see figure). A steel cube is inserted into the recess without a clearance, and compressed by a force P of 700 kg, as shown in the figure. Assuming the slab incompressible determine all three principal stresses in the cube.

Answer: $\sigma_1 = \sigma_2 = -300$ kg/cm^2; $\sigma_3 = -700$ kg/cm^2.



PROBLEM 2.25

2.26. A thick steel slab has a square recess (in plan view), with a depth of 10 mm and transverse dimensions of 10·001 × 10·001 mm^2.

A cube of similar steel, with dimensions of $10 \times 10 \times 10$ mm^3, is inserted into this recess and compressed by a force P of 1500 kg on the free face. Assuming that the slab is not deformable, determine all three principal stresses in the cube.

Solution. The compressive stresses in the free surface, and in the surface opposite to it, will be the greatest. We shall therefore denote them by σ_3.

They are

$$\sigma_3 = \frac{P}{F} = -\frac{1500}{1} = -1500 \text{ kg/cm}^2.$$

In view of the symmetry of the recess and cube the two remaining principal stresses are equal to each other: $\sigma_1 = \sigma_2$. Also identical are the strains ε_1 and ε_2. They are

$$\varepsilon_1 = \varepsilon_2 = \frac{10{\cdot}001 - 10}{10} = 1 \times 10^{-4}.$$

The magnitude of strain in a three-dimensional state of stress is determined by the formula

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \mu(\sigma_2 + \sigma_3)).$$

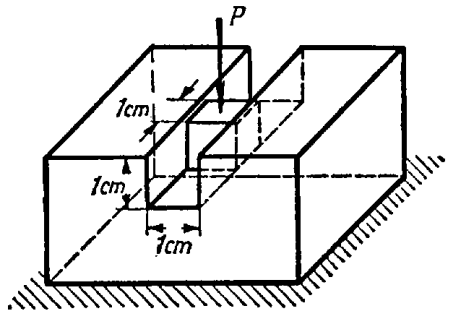
Putting $\sigma_2 = \sigma_1$ we solve this equation for σ_1, and substitute the values for ε_1 and ε_3:

$$\sigma_1 = \sigma_2 = \frac{E\varepsilon_1 + \mu\sigma_3}{1 - \mu} = \frac{(2 \times 10^6 \times 10^{-4}) - (0{\cdot}3 \times 1500)}{1 - 0{\cdot}3} =$$

$$= -357 \text{ kg/cm}^2.$$

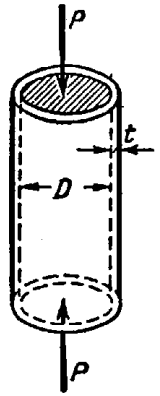
2.27. A square hole, 1 cm side, is made to pass through a thick steel slab. An aluminium cube with dimensions of $1 \times 1 \times 1$ cm^3 is inserted into this recess without clearances, and compressed as indicated in the figure by a force P of 600 kg. Assuming that the slab is incompressible determine all three principal stresses in the cube. For aluminium μ is 0·33.

Answer: $\sigma_1 = 0$; $\sigma_2 = -198$ kg/cm², $\sigma_3 = -600$ kg/cm².



PROBLEM 2.27

2.28. A solid copper cylinder, with diameter D of 50 mm, is enclosed by a steel tube with thickness t of 1 mm (see figure). The cylinder is compressed by forces, P, of 20 t. Determine the tensile stresses in the cylinder if Poisson's ratio, μ, for copper is 0·32. Neglect the friction between the cylinder and the tube.



PROBLEM 2.28

Solution. We shall cut out from the cylinder a cube with edges equal to unity. The stresses in its top and bottom faces are given by

$$\sigma_3 = \frac{P}{F_{\text{cyl.}}} = \frac{4P}{\pi D^2} = \frac{4 \times 20{,}000}{3{\cdot}14 \times 5^2} = -1015 \text{ kg/cm}^2.$$

The stresses in the side faces will also be compressive and, in view of the symmetry, equal. We shall denote them by $\sigma_1 = \sigma_2$

$= -q$. These stresses will be equal to the pressure on the inner surface of the tube. It is known that, due to an internal pressure q, the tensile stresses in a cylinder in the direction perpendicular to the generator are given by

$$\sigma = \frac{qD}{2t},$$

whence

$$q = \frac{2t\sigma}{D}.$$

In the tube there will be no stresses parallel to the generator, since in this direction there is no load (the friction force between the bar and tube is not taken into consideration).

For the determination of the stresses σ we shall consider the deformations. The strain of the copper cylinder is equal to the lateral strain of the cube cut out from it:

$$\varepsilon_{\text{cyl.}} = \varepsilon_{\text{cube}} = \frac{1}{E_c}(-q - \mu(-q + \sigma_3)).$$

The strain of the diameter of the tube, equal to the strain of its circumference

$$\varepsilon_{\text{tube}} = \frac{\sigma}{E_s},$$

is equal to the transverse strain of the cylinder:

$$\varepsilon_{\text{cyl.}} = \varepsilon_{\text{tube}}.$$

We shall substitute in this equation the values for $\varepsilon_{\text{cyl.}}$ and $\varepsilon_{\text{tube}}$:

$$\frac{1}{E_c}(-q - \mu(-q + \sigma_3)) = \frac{\sigma}{E_s}$$

and replace q by its expression in terms of σ:

$$\frac{1}{E_c}\left[-\frac{2t\sigma}{D} - \mu\left(-\frac{2t\sigma}{D} + \sigma_3\right)\right] = \frac{\sigma}{E_s}.$$

Solving this equation for σ and substituting the numerical values for the remaining quantities, we shall determine the stress σ:

$$\sigma = \frac{-\mu\sigma_3}{2\frac{t}{D}(1-\mu)+\frac{E_c}{E_s}} = \frac{0{\cdot}32 \times 1015}{2 \times \frac{1}{50}(1-0{\cdot}23)+\frac{1}{2}} = 616 \text{ kg/cm}^2.$$

2.29. A concrete cylinder with diameter D of 30 cm is enclosed by a steel tube with thickness t of 2 mm, and compressed by a force P of 15 t. Find the principal stresses for a cube cut out from the cylinder. The modulus of elasticity E for concrete is to be taken as 14×10^4 kg/cm^2, and its Poisson's ratio μ as 0·18. Solve the same problem, assuming that the wall thickness is great and that the deformation of the tube may be neglected.

Answer: (1) $\sigma_1 = \sigma_2 = -0{\cdot}63$ kg/cm^2;
(2) $\sigma_1 = \sigma_2 = -4{\cdot}7$ kg/cm^2.

2.30. A cylinder 5 cm in diameter and 5 cm long is made of aluminium. It is enclosed by a steel tube 5 cm in internal diameter, and 5·6 cm in external diameter. The cylinder is compressed axially by stresses of 1000 kg/cm^2. (1) What will be the axial deformation of the cylinder? (2) How will the deformation of the cylinder change, if the internal diameter of the tube is 5·005 cm? Take μ as 0·35 for aluminium.

Answer: (1) $\Delta l = -0{\cdot}067$ mm; (2) $\Delta l = -0{\cdot}0713$ mm.

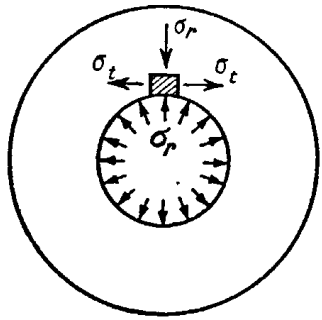
2.31. During firing the gases of the explosive cause an internal pressure of 3500 kg/cm^2 in the barrel of a gun. The maximum radial compressive stress σ_r in the material is consequently also 3500 kg/cm^2. The maximum tangential (tensile) stress σ_t equals 500 kg/cm^2, and occurs at the same place as the radial stresses stated above. The third principal stress may be assumed to be zero. What are the maximum shear stresses ans in what planed do they act? Calculate the effective stresses in accordance with the 4th theory of strength.

Solution. As is well known the maximum stress is determined by the formula

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

In this case $\sigma_1 = 5500$ kg/cm², $\sigma_2 = 0$ and $\sigma_3 = -3500$ kg/cm². Consequently

$$\tau_{max} = \frac{5500 + 3500}{2} = 4500 \text{ kg/cm}^2.$$



PROBLEM 2.31

The planes in which they occur form an angle of 45° with the planes in which σ_1 and σ_3 occur. In our case these will be the planes inclined at an angle of 45° to any radius of the inner surface of the gun barrel, and will be perpendicular to the transverse section of the barrel.

In accordance with the 4th theory of strength the effective stresses are determined by the formula

$$\sigma_{r_4} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} =$$

$$= \frac{1}{\sqrt{2}} \sqrt{(5500 - 0)^2 + (5500 + 3500)^2 + (0 + 3500)^2} =$$

$$= 7860 \text{ kg/cm}^2.$$

2.32. If in the barrel of the gun σ_t is 5500 kg/cm², σ_r is −3500 kg/cm², and the third principal stress acting perpendicular to the plane of the figure (along the barrel of the gun) is tensile and equal to 4200 kg/cm²—what is then the maximum shear stress and in what plane does it occur? Calculate the effective stresses in accordance with the third theory of strength.

Answer: $\tau_{max} = 4500$ kg/cm²; $\sigma_{r_3} = 9000$ kg/cm².

2.33. Calculate the effective stresses in accordance with the second, third, and fourth theories of strength, if the principal stresses have the following values, and μ is 0·3:

(a) $\sigma_1 = 1200$ kg/cm², $\sigma_2 = 1000$ kg/cm², $\sigma_3 = 800$ kg/cm²;
(b) $\sigma_1 = 1200$ kg/cm², $\sigma_2 = 1000$ kg/cm², $\sigma_3 = -800$ kg/cm²;
(c) $\sigma_1 = 1200$ kg/cm², $\sigma_3 = -1000$ kg/cm², $\sigma_2 = 800$ kg/cm²;
(d) $\sigma_1 = 1200$ kg/cm², $\sigma_3 = -1000$ kg/cm², $\sigma_2 = -800$ kg/cm²;
(e) $\sigma_3 = -1200$ kg/cm², $\sigma_1 = 1000$ kg/cm², $\sigma_2 = 800$ kg/cm²;
(f) $\sigma_3 = -1200$ kg/cm², $\sigma_1 = 1000$ kg/cm², $\sigma_2 = -800$ kg/cm²;
(g) $\sigma_3 = -1200$ kg/cm², $\sigma_2 = -1000$ kg/cm², $\sigma_1 = 800$ kg/cm²;
(h) $\sigma_3 = -1200$ kg/cm², $\sigma_2 = -1000$ kg/cm², $\sigma_1 = -800$ kg/cm².

Answer:

(a) $\sigma_{r_2} = 660$ kg/cm², $\sigma_{r_3} = 400$ kg/cm², $\sigma_{r_4} = 346$ kg/cm²;
(b) $\sigma_{r_2} = 1460$ kg/cm², $\sigma_{r_3} = 2000$ kg/cm², $\sigma_{r_4} = 1910$ kg/cm²;
(c) $\sigma_{r_2} = 1600$ kg/cm², $\sigma_{r_3} = 2200$ kg/cm², $\sigma_{r_4} = 2030$ kg/cm²;
(d) $\sigma_{r_2} = 1740$ kg/cm², $\sigma_{r_3} = 2200$ kg/cm², $\sigma_{r_4} = 2110$ kg/cm²;
(e) $\sigma_{r_2} = 1740$ kg/cm², $\sigma_{r_3} = 2200$ kg/cm², $\sigma_{r_4} = 2110$ kg/cm²;
(f) $\sigma_{r_2} = 1600$ kg/cm², $\sigma_{r_3} = 2200$ kg/cm², $\sigma_{r_4} = 2030$ kg/cm²;
(g) $\sigma_{r_2} = 1460$ kg/cm², $\sigma_{r_3} = 2000$ kg/cm², $\sigma_{r_4} = 1910$ kg/cm²;
(h) $\sigma_{r_2} = 660$ kg/cm², $\sigma_{r_3} = 400$ kg/cm², $\sigma_{r_4} = 346$ kg/cm².

7. CALCULATION OF THIN-WALLED VESSELS

2.34. A cylindrical boiler with diameter D of 2 m and wall thickness t of 10 mm is under a working pressure q, equal to 10 atm. The allowable stress $[\sigma]$ of the material of the boiler walls is 900 kg/cm². Check the strength of the boiler walls on the basis of the fourth theory of strength.

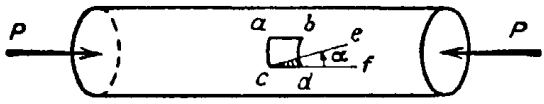
Answer: 866 kg/cm² < 900 kg/cm².

2.35. A cylindrical boiler 1·8 m in diameter has a plate thickness of 20 mm. It is subjected to a working pressure of 14 atm. Determine the maximum shear stress in the boiler wall. In what planes does it occur? Calculate the effective stress in accordance with the second theory of strength, if the material of the boiler walls is steel.

Answer: $\tau_{max} = 315$ kg/cm²; $\sigma_{r_2} = 535$ kg/cm².

2.36. A rectangular element *abcd* is cut out from the wall of a cylindrical steel tank (see figure); one of its sides is parallel to the generator of the cylinder. The diameter D of the cylinder is 100 cm, the wall thickness t is 10 mm, the internal pressure q is 20 atm; the cylinder is compressed axially by forces P of 50 t on the end plates.

Determine the normal and shear stresses in section *ce* perpendicular to the plane of the element, and forming an angle of 30° with the direction of generator *cf* of the cylinder. Calculate the effective stresses in accordance with the second, third, and fourth theories of strength.



PROBLEM 2.36

Solution. To find the answer it will be necessary to determine the principal stresses. They will act in planes parallel and perpendicular to the generator of the cylinder. In the first plane (parallel to the generator) the normal stress σ_x is

$$\sigma_x = + \frac{qD}{2t} = + \frac{20 \times 100}{2 \times 1} = +1000 \text{ kg/cm}^2;$$

in the second plane (perpendicular to the generator)

$$\sigma_y = + \frac{qD}{4t} - \frac{P}{\pi Dt} = + \frac{20 \times 100}{4 \times 1} - \frac{50{,}000}{\pi \times 100 \times 1} =$$

$$= + 500 - 159 = +341 \text{ kg/cm}^2.$$

The principal stresses are thus

$$\sigma_1 = +1000 \text{ kg/cm}^2, \qquad \sigma_2 = +341 \text{ kg/cm}^2, \qquad \sigma_3 = 0.$$

The normal and shear stresses in an inclined plane are respectively

$$\sigma_\alpha = \sigma_1 \cos^2 \alpha + \sigma_2 \sin^2 \alpha, \qquad \text{and} \qquad \tau_\alpha = \frac{\sigma_1 - \sigma_2}{2} \sin 2\alpha,$$

where α is the angle between the normal of the plane and the direction of the maximum principal stress σ_1. In the case being considered σ_1 is directed perpendicularly to the generator of the cylinder. The angle between the inclined plane and the generator is 30°; consequently the angle between the normal of the plane and the direction of σ_1 is 60°. Then

$$\sigma_\alpha = 1000 \cos^2 60° + 341 \sin^2 60° = 506 \text{ kg/cm}^2,$$

$$\tau_\alpha = \frac{1000 - 341}{2} \sin 120° = 285 \text{ kg/cm}^2.$$

The formulae for calculating the effective stresses have the form:

By the second theory of strength $\sigma_{r_2} = [\sigma_1 - \mu(\sigma_2 + \sigma_3)]$;
By the third theory of strength $\sigma_{r_3} = [\sigma_1 - \sigma_3]$;
By the fourth theory of strength $\sigma_{r_4} =$

$$= \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3}.$$

In our case:

$\sigma_{r_2} = 1000 - (0{\cdot}3 \times 341) = 898 \text{ kg/cm}^2$;
$\sigma_{r_3} = 1000 - 0 = 1000 \text{ kg/cm}^2$;
$\sigma_{r_3} = \sqrt{1000^2 + 341^2 - (1000 \times 341)} = 881 \text{ kg/cm}^2$.

2.37. A rectangular element is cut out from the wall of a cylindrical boiler 1·5 m in diameter subjected to a working pressure of 14 atm; one of its sides is parallel to the generator of the cylinder. The wall thickness of the boiler is 12 mm. Find the normal and shear stresses in a plane perpendicular to the plane of the element, and inclined at an angle of 45° to the generator. Calculate the effective stresses in accordance with the fourth theory of strength.

Answer: $\sigma_\alpha = 656 \text{kg/cm}^2$; $\tau_\alpha = 219 \text{kg/cm}^2$; $\sigma_{r_4} = 758 \text{kg/cm}^2$.

2.38. Determine the required wall thickness of a cast iron water pipe 100 cm in diameter, subjected to a 120 m head of water. The allowable tensile stress for cast iron is to be taken as 200 kg/cm².

Answer: 30 mm.

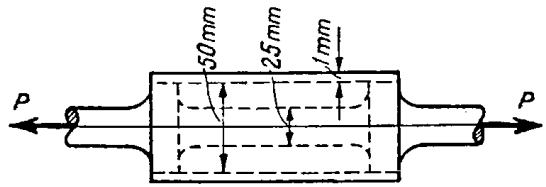
2.39. A steel water pipe 60 cm in diameter works under a 120 m head of water. What must be the wall thickness of the pipe, if the

allowable stress is taken to be 800 kg/cm², and the possible reduction in the wall thickness due to corrosion is assumed to be 2·5 mm?

Answer: 7 mm.

2.40. The steel tube represented in the figure is welded to the flanged parts of a steel bar. The tube is subjected to internal pressure, causing in it (away from the ends of the tube) a tensile stress of 1400 kg/cm² perpendicular to the generator. At the same time the bar is pulled by an axial force P of 4 t. The bar and tube are elongated by equal amounts. Determine the longitudinal tensile stresses in the middle of the bar and tube. Assume that μ is 0·25.

Answer: $\sigma_{bar} = 655$ kg/cm²; $\sigma_{tube} = 1005$ kg/cm².



PROBLEM 2.40

2.41. What internal pressure is allowable in a vessel formed by a hollow copper cylinder (internal diameter 50 cm, wall thickness 0·4 cm) and a steel cylinder with a wall thickness of 0·2 cm, closely enveloping the copper cylinder over its whole length, if the maximum allowable stress is 400 kg/cm² for copper and 1600 kg/cm² for steel? In the calculation neglect the stresses parallel to the generator.

Solution. The diametral, and consequently the circumferential strains of both cylinders will be equal. In such a case the ratio of the stresses in them will be equal to the ratio of the elastic moduli:

$$\frac{\sigma_s}{\sigma_c} = \frac{E_s}{E_c} = 2.$$

In determining the maximum allowable pressure it will be necessary to base them on the copper cylinder—since, if we determine the pressure at which the stresses in the steel cylinder equal the

allowable 1600 kg/cm², the stresses in the copper cylinder would reach 800 kg/cm² (which is not permissible, as they exceed 400 kg/cm²).

We imagine both cylinders cut in two, and set up the equation of equilibrium for one of the halves, putting $\sigma_s = 2\sigma_c$:

$$2\sigma_c t_c l + (2 \times 2\sigma_c t_s l) - qdl = 0,$$

whence

$$q = \frac{2\sigma_c(t_c + 2t_s)}{d}.$$

We shall put $\sigma_c = [\sigma_c]$; then

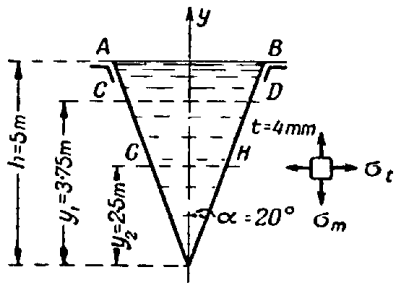
$$q = \frac{2[\sigma_c]\,(t_c + 2t_s)}{d} = \frac{2 \times 400\,[0{\cdot}4 + (2 \times 0{\cdot}2)]}{50} =$$

$$= 12{\cdot}8 \text{ kg/cm}^2 \approx 13 \text{ atm}.$$

2.42. Determine the stress in the wall of a steel cylinder enclosed by an aluminium casing. The inside diameter d_1 of the cylinder is 160 mm, the wall thickness t_1 is 3 mm; the thickness t_2 of the aluminium casing is 6·5 mm. The pressure, q, of the gases in the cylinder is 33 kg/cm². For the calculation imagine a ring cut out of the cylinder, having a width equal to unity, and assume that no stresses exist in the axial direction of the cylinder.

Answer: 500 kg/cm².

2.43. Determine the maximum value of the normal stress in a conical tank (with the vertex angle $2\alpha = 40°$) suspended along the circumference at section AB, and filled with water to a height h of 5 m (see figure). The wall thickness is 4 mm.



PROBLEM 2.43

Solution. The tensile stresses occurring in the wall of the tank filled with liquid are σ_m (in the direction of a generator) and σ_t (in the tangential direction). For the calculation of these stresses we have the formulae

$$\sigma_m = \frac{\gamma \tan \alpha}{2t \cos \alpha}\left(h - \frac{2}{3}y\right)y \quad \text{and} \quad \sigma_t = \frac{\gamma \tan \alpha}{t \cos \alpha}(h - y)\,y,$$

where γ is the specific weight of the liquid, and y is a distance along the cone axis from its vertex to the plane in which the stresses are to be determined. The stresses σ_m and σ_t vary parabolically, and are dependent on the distance y. We shall determine the position of the planes in which the maximum values of these occur. To do this we shall equate to zero the derivatives (with respect to y) of the expressions for σ_m and σ_t. We have

$$\frac{d\sigma_m}{dy} = \frac{\gamma \tan \alpha}{2t \cos \alpha}\left(h - \frac{4}{3}y\right) = 0,$$

whence

$$y_1 = \frac{3}{4}h = \frac{3}{4} \times 5 = 3{\cdot}75\ \text{m} \qquad \text{(on level CD)},$$

and

$$\frac{d\sigma_t}{dy} = \frac{\gamma \tan \alpha}{t \cos \alpha}(h - 2y) = 0,$$

whence

$$y_2 = \frac{h}{2} = \frac{5}{2} = 2{\cdot}5\ \text{m} \qquad \text{(on level GH)}.$$

Substituting the values for y_1 and y_2, and for the remaining quantities of the problem in the expressions for σ_m and σ_t, we find

$$\max \sigma_m = \frac{\gamma \tan \alpha}{2t \cos \alpha}\left(h - \frac{2}{3} \times \frac{3}{4}h\right)\frac{3}{4}h =$$

$$= \frac{3\gamma h^2 \tan \alpha}{16\ t \cos \alpha} = \frac{3 \times 0{\cdot}001 \times 500^2 \times 0{\cdot}364}{16 \times 0{\cdot}4 \times 0{\cdot}940} =$$

$$= 45{\cdot}4\ \text{kg/cm}^2$$

and

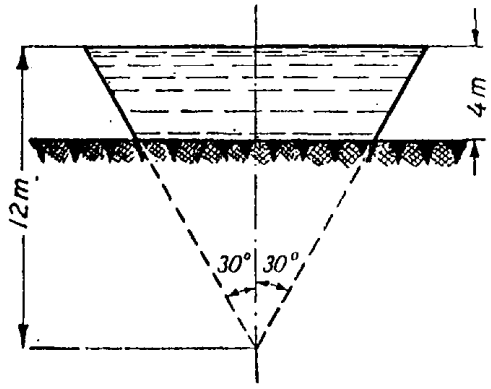
$$\max \sigma_t = \frac{\gamma \tan \alpha}{2t \cos \alpha}\left(h - \frac{h}{2}\right)\frac{h}{2} = \frac{\gamma h^2 \tan \alpha}{4t \cos \alpha} =$$

$$= \frac{0{\cdot}001 \times 500^2 \times 0{\cdot}364}{4 \times 0{\cdot}4 \times 0{\cdot}940} = 60{\cdot}5 \text{ kg/cm}^2.$$

The maximum normal stress $\sigma_{max} = \max \sigma_t = 60{\cdot}5$ kg/cm^2.

2.44. A tank in the form of a truncated cone (see figure) is supported on its smaller end. It is filled with a liquid of specific weight of 0·9 t/m^3. The wall thickness of the tank is 8 mm. Determine the maximum normal stresses in the meridional and tangential directions.

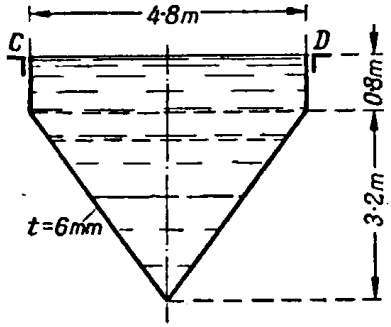
Answer: $\sigma_m = 70$ kg/cm^2 (compression); $\sigma_t = 240$ kg/cm^2.



PROBLEM 2.44

2.45. A tank consists of two parts (see figure): an upper cylindrical part 4·8 m in diameter and 80 cm in height, and a lower conical part having a height of 3·2 m and a wall thickness of 6 mm. The tank is suspended along the circumference at section CD, and filled with petrol of specific weight of 0·72 t/m^3. Determine the maximum normal stresses in the meridional and tangential directions in the wall of the conical part of the tank.

Answer: max $\sigma_m = 33{\cdot}8$ kg/cm^2; max $\sigma_t = 45$ kg/cm^2.



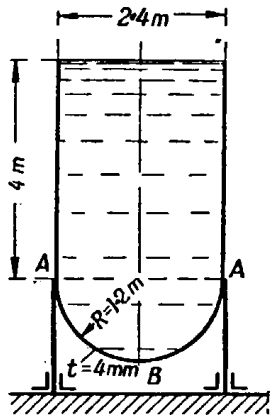
PROBLEM 2.45

2.46. A spherical steel vessel having a diameter of 60 cm is subjected to an internal, evenly distributed pressure of 24 atm. Determine the required wall thickness of the vessel, if the allowable stress is 1200 kg/cm^2.

Answer: 3 mm.

2.47. A cylindrical tank 2·4 m in diameter, with a bottom in the form of a hemisphere is filled with water to a height of 4 m (see figure). The wall thickness of the cylindrical part and the bottom of the tank is 4 mm. Determine the maximum normal stresses in the wall of the cylindrical part and the bottom of the tank.

Answer: max $\sigma_{cyl.} = 96$ kg/cm^2; max $\sigma_{bottom} = 62{\cdot}4$ kg/cm^2.



PROBLEM 2.47

8. CALCULATION OF THICK-WALLED VESSELS

2.48. A long concrete pipe with an internal diameter of 1 m is laid at a depth H, 35 m below the water surface. Assuming the water pressure is distributed uniformly over the surface of the pipe, determine its required wall thickness—proceeding from the limiting strength condition according to the theory of maximum stresses, and according to the theory of maximum strains. Take the allowable stress for the concrete as 15 kg/cm^2 in compression, and Poisson's ratio μ as 0·16.

Solution. The pipe is subjected only to the external pressure $p_1 = \gamma H = 1{\cdot}0 \times 35 = 35$ t/m^2 $= 3{\cdot}5$ kg/cm^2 uniformly distributed over its surface. For the calculation of the normal stresses in the radial and tangential directions we have the formulae

$$\sigma_r = -\frac{p_1 r_1^2}{r_1^2 - r_2^2}\left(1 - \frac{r_2^2}{r^2}\right) \quad \text{and} \quad \sigma_t = -\frac{p_1 r_1^2}{r_1^2 - r_2^2}\left(1 + \frac{r_2^2}{r^2}\right),$$

where r_1 is the radius of the external and r_2 the radius of the internal cylindrical surface of the pipe; r is the distance from the axis of the pipe to any area in the cross-section. As the pipe is long, then for a section far from its ends the strain in the direction of the pipe axis (ε_z) should be assumed to be zero, i.e.

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \mu(\sigma_r + \sigma_t)\right) = 0,$$

whence

$$\sigma_z = \mu(\sigma_r + \sigma_t) = -2\mu p_1 \frac{r_1^2}{r_1^2 - r_2^2} = \text{const.}$$

The stresses σ_r, σ_t and σ_z along the whole thickness of the pipe will be compressive; the greatest of these, in absolute value, will be the tangential stress σ_t. This stress attains a maximum at the inner surface of the pipe. Here at $r = r_2$

$$\sigma_t = -\frac{2p_1 r_1^2}{r_1^2 - r_2^2};$$

thereby $\sigma_r = 0$.

The limiting strength condition according to the theory of maximum normal stresses has the form

$$|\sigma_3| = |\max \sigma_t| = \frac{2p_1 r_1^2}{r_1^2 - r_2^2} \leq [\sigma_{\text{comp.}}],$$

whence

$$r_1 \geq \frac{r_2}{\sqrt{1 - \dfrac{2p_1}{[\sigma_{\text{comp.}}]}}}.$$

Substituting the data given with the problem we find that

$$r_1 \geq \frac{50}{\sqrt{1 - \dfrac{2 \times 3{\cdot}5}{15}}} = 68{\cdot}5 \text{ cm},$$

i.e. the wall of the pipe must have the thickness $\delta = r_1 - r_2 = 68{\cdot}5 - 50 = 18{\cdot}5$ cm.

The limiting strength condition according to the theory of maximum strains, for the material at the inner surface of the pipe, is of the form

$$|\sigma_3 - \mu(\sigma_1 + \sigma_2)| = |\sigma_t - \mu(\sigma_r + \sigma_z)| = |\sigma_t - \mu\sigma_z| =$$

$$= \frac{2p_1 r_1^2}{r_1^2 - r_2^2} - 2\mu^2 \frac{p_1 r_1^2}{r_1^2 - r_2^2} = \frac{2(1 - \mu^2)\, p_1 r_1^2}{r_1^2 - r_2^2} \leq [\sigma_{\text{comp.}}],$$

whence

$$r_1 \geq \frac{r_2}{\sqrt{1 - \dfrac{2p_1}{[\sigma_{\text{comp.}}]}(1 - \mu^2)}} = \frac{50}{\sqrt{1 - \dfrac{2 \times 3{\cdot}5}{15}(1 - 0{\cdot}16^2)}},$$

$$= 68 \text{ cm},$$

i.e. the wall of the pipe should be 18 cm thick.

2.49. A long steel pipe having an internal diameter of 4 cm and wall thickness of 5 mm is subjected to an internal uniformly distributed pressure of 90 kg/cm^2. Assuming the strain in the axial direction of the pipe is zero determine the maximum effective

stresses in the pipe, in accordance with the theory of the maximum shear strain energy.

Answer: 438 kg/cm^2.

2.50. A hollow steel cylinder with end covers, having an internal diameter of 150 mm and a wall thickness of 25 mm is subjected to an internal uniformly distributed pressure of 400 atm. Assuming the normal stresses in the axial direction of the cylinder are uniformly distributed over its transverse section, determine the value of the maximum effective stress in the wall of the cylinder: (a) on the basis of the theory of maximum shear stresses, and (b) on the basis of the theory of maximum shear strain energy.

Answer: (a) 1830 kg/cm^2; (b) 1585 kg/cm^2.

2.51. A long hollow steel cylinder having an external diameter of 200 mm is subjected to an internal, uniformly distributed pressure of 350 atm. Assuming the strain in the direction of the cylinder axis is zero, and proceeding from the limiting strength condition (1) according to the theory of maximum shear stresses, and (2) according to the theory of maximum shear strain energy determine the required wall thickness of the cylinder, if the allowable stress $[\sigma]$ is 2000 kg/cm^2.

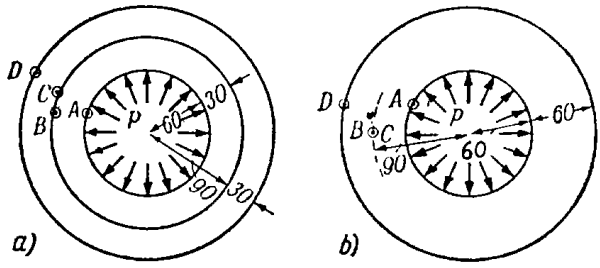
Answer: (1) 19·4 mm; (2) 16·8 mm.

2.52. A long steel sleeve having an external diameter of 240 mm is placed, while hot, on to a solid steel shaft 160 mm in diameter. The internal diameter of the sleeve is made 0·2 mm smaller than the diameter of the shaft. Determine the maximum tensile and compressive stresses in the material of the sleeve and shaft.

Answer: In the sleeve: $\max \sigma_{\text{tens.}} = 1805$ kg/cm^2; $\max \sigma_{\text{comp.}} = 695$ kg/cm^2. In the shaft: $\max \sigma_{\text{comp.}} = 695$ kg/cm^2 = const.

2.53. A compound cylinder is made up of two steel pipes: the first one has an internal diameter of 120 mm and a wall thickness of 30 mm; the second one has an external diameter of 240 mm (see figure (a)). The second pipe is heated up and placed over the first one. Before the operation the internal diameter of the second pipe was 0·135 mm smaller than the external diameter of the first pipe.

The compound cylinder is subjected to an internal, uniformly distributed pressure p.



PROBLEM 2.53

Applying the theory of maximum shear stresses determine the maximum allowable value of the pressure p, if the allowable stress for the pipe material is 3000 kg/cm².

Compare the maximum allowable value of p for the compound cylinder with that for a hollow cylinder of the same steel, having an internal diameter of 120 mm and wall thickness of 60 mm (see figure (b)). Construct diagrams showing the distribution of the normal stresses across the wall thickness of both cylinders for the maximum value of the pressure p.

Answer:

	$[p]$ kg/cm²	Stresses (kg/cm²) at points				
		Stresses σ_r and σ_t	A	B Inner pipe	C Outer pipe	D
Compound cylinder	1453	σ_r	−1453	− 620	− 620	0
		σ_t	+1547	+ 713	+2213	+1594
Single cylinder	1125	σ_r	−1125	− 292		0
		σ_t	+1875	+1042		+ 750

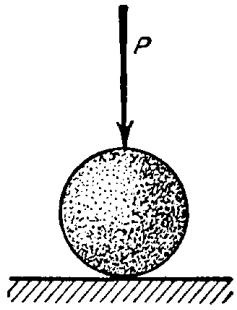
2.54. A spherical hollow steel vessel with an internal diameter of 50 cm is intended to store gas compressed under a pressure of 800 atm. Proceeding from the condition of limiting strength according to the theory of maximum shear strain energy, determine the required wall thickness if the allowable stress is 2500 kg/cm².

What should be the wall thickness of a closed cylindrical vessel made of the same steel and having the same internal diameter?

Answer: 61 mm; 125 mm.

9. CONTACT STRESSES*

2.55. Determine the value of the maximum contact stress at the initial instant when the steel ball of a Brinell testing machine is being pressed into a plane component under test (see figure). The diameter of the ball is 10 mm, the force P is 1 kg. The ball and the component are of steel.



PROBLEM 2.55

Solution. The formula for the maximum stress for the given type of contact is given in the book *Strength of Materials* by N. M. Belyayev (Table 11 †). It is of the form

$$\sigma_{\max} = 0{\cdot}388 \sqrt[3]{PE^2 \frac{1}{R^2}}.$$

In this case $P = 1$ kg, $E = 2 \times 10^4$ kg/mm^2 and $R = 5$ mm; consequently

$$\sigma_{\max} = 0{\cdot}388 \sqrt[3]{\frac{(2 \times 10^4)^2}{5^2}} = 98 \text{ kg/mm}^2.$$

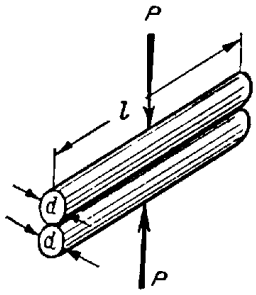
* The solution of a number of the most important problems on contact stresses is due to N. M. Belyayev, I. Ya. Shtaerman, G. N. Savin, L. A. Galin and other Soviet scientists.

† See Appendix B, Section 6.

We see that already at the initial instant very large stresses occur in the component being tested. At an increase in the load they pass the yield point, and a plastic deformation (indentation) is formed.

2.56. The transmission of a force P of 10 t (the resultant of a uniform pressure) is effected by means of two contacting steel cylinders, of diameter d and length l of 200 mm (see figure). Determine the diameter d of the cylinders, if the maximum stress on the contact surface is not to exceed 100 kg/cm.

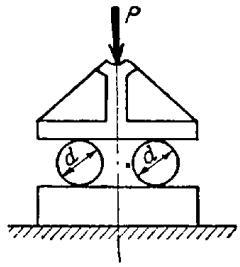
Answer: 70 mm.



PROBLEM 2.56

2.57. A movable articulated support for a bridge girder has two cylindrical rollers, each with a diameter d, of 100 mm and a length, l, of 300 mm. A thrust P on these rollers, being equally distributed between them, is 100 t (see figure). Determine the maximum principal compressive stresses in the supporting surfaces of the rollers.

Answer: $-107{\cdot}5$ kg/mm^2.



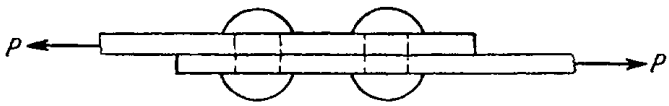
PROBLEM 2.57

CHAPTER 3

SHEAR AND TORSION

10. SHEAR

3.1. Determine the required number of rivets 20 mm in diameter, for a lap joint of two plates 8 mm and 10 mm thick (see figure). The tensile force P on the joint is 20 t. The allowable stresses are: $[\tau] = 1400$ kg/cm^2 in shear and $[\sigma_c] = 3200$ kg/cm^2 in compression.



PROBLEM 3.1

Solution. From the equation of the limiting shear stress the required number of rivets

$$n \geq \frac{P}{\frac{\pi d^2}{4}[\tau]} = \frac{20{,}000}{\pi \times \frac{2^2}{4} \times 1400} = 4{\cdot}53 \approx 5.$$

From the equation of the limiting compressive stress the required number of rivets

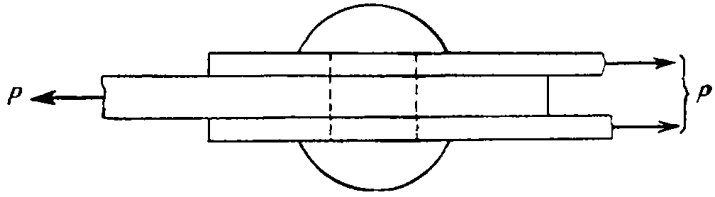
$$n \geq \frac{P}{td[\sigma_c]} = \frac{20{,}000}{0{\cdot}8 \times 2 \times 3200} = 3{\cdot}9 \approx 4.$$

Thus it will be necessary to provide five rivets.

3.2. Determine the number of rivets 20 mm in diameter for joining two plates 5 mm thick to a third plate 12 mm thick (see figure). The

tensile force, P, on the joint is 18 t. The allowable stresses are: $[\tau] = 1000$ kg/cm^2 and $[\sigma_c] = 2800$ kg/cm^2.

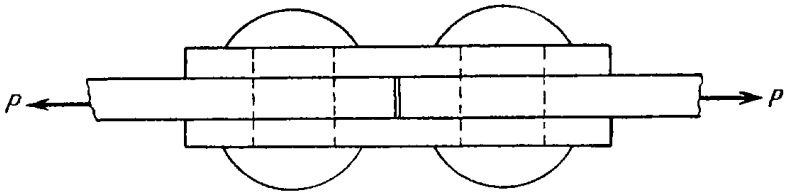
Answer: Four rivets.



PROBLEM 3.2

3.3. Determine the required number of rivets, 17 mm in diameter, for a butt joint of two plates with two straps (see figure). The tensile force, P, is 30 t. The thickness of the plates is 10 mm each; that of the straps is 6 mm each. The allowable stresses are: $[\tau] = 1000$ kg/cm^2 in shear and $[\sigma_c] = 2800$ kg/cm^2 in compression.

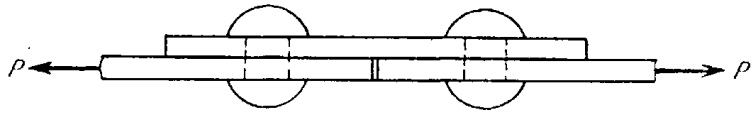
Answer: Seven rivets in each half of the joint.



PROBLEM 3.3

3.4. Two plates are joined by means of a single strap as shown in the figure. The thickness of the plates and strap is 10 mm each. Determine the required number of 17 mm diameter rivets, if the allowable stresses are 1400 kg/cm^2 in shear ($= [\tau]$) and 3200 kg/cm^2 in compression ($= [\sigma_c]$). The force P pulling the joint equals 24 t.

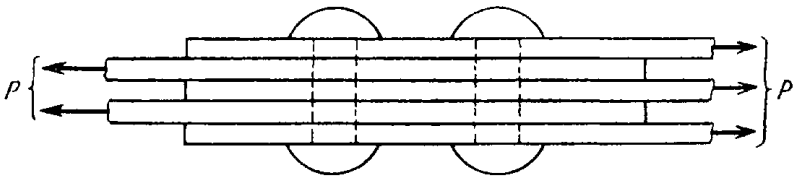
Answer: Eight rivets in each half of the joint.



PROBLEM 3.4

3.5. Two plates, each 10 mm thick, are joined to three plates, each 8 mm thick, by means of 20 mm diameter rivets in the manner indicated in the figure. The joint is subjected to a tensile force P, of 28 t. Determine the required number of rivets, if the allowable stresses $[\tau]$ and $[\sigma_c]$ are 1000 kg/cm^2 and 2800 kg/cm^2 respectively.

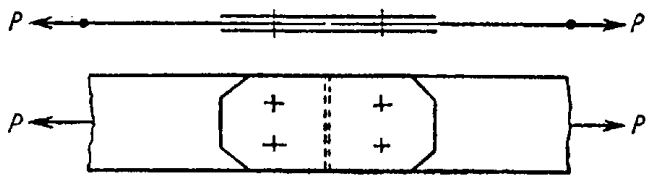
Answer: Three rivets.



PROBLEM 3.5

3.6. A plate with a transverse section of 12 $\times$ 150 mm^2 has a butt joint (see figure) with two 6 mm thick straps. In the joint, rivets 20 mm in diameter are used. The tensile force P is 12 t. Determine the shear and bearing stresses in the rivets, and the tensile stresses in the plates and straps.

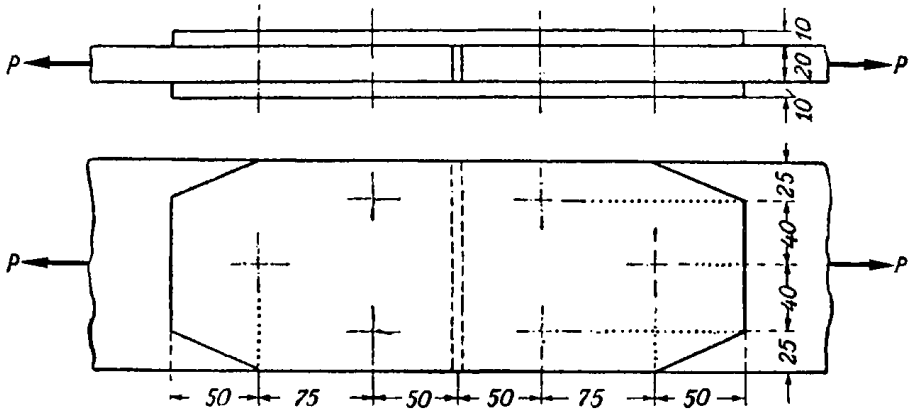
Answer: In kg/cm^2: 955; 2500; 909; 909.



PROBLEM 3.6

3.7. In a joint of two plates, rivets 26 mm in diameter are used. The connection is made as shown in the figure. The allowable stresses $[\tau]$, $[\sigma_c]$, and $[\sigma]$ are 1000 kg/cm^2, 2800 kg/cm^2, and 1600 kg/cm^2 respectively. Determine the maximum allowable tensile force on this joint.

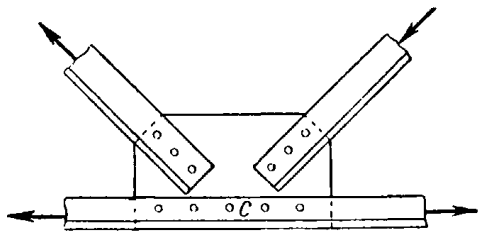
Answer: 25 t.



PROBLEM 3.7

3.8. The diagonal members of a girder are subjected to forces of 5·25 t. Each of them is fastened at joint C to a gusset plate 10 mm thick by means of three rivets. The cross-section of the diagonal members is that of a 75 mm $\times$ 75 mm $\times$ 75 mm angle. Determine the shear and bearing stresses in the rivets.

Answer: $\tau = 556$ kg/cm^2; $\sigma_c = 1090$ kg/cm^2.

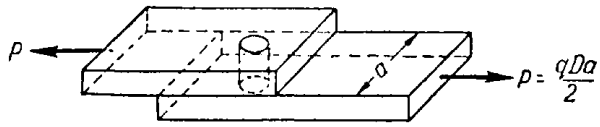


PROBLEM 3.8

3.9. A cylindrical boiler 150 cm in diameter is subjected to an internal pressure of 4 atm. The longitudinal joint is made as a lap joint with a single row of rivets. The thickness of the plate is 10 mm, the diameter of the rivets is 20 mm and the pitch of the rivets is 75 mm. Determine the shear stress, the bearing stress and

the tensile stress in the joint. Determine also the limiting allowable pressure in the boiler, if the allowable stresses are:

$$[\sigma] = 1000 \text{ kg/cm}^2, \ [\tau] = 720 \text{ kg/cm}^2 \text{ and } [\sigma_c] = 1600 \text{ kg/cm}^2.$$



PROBLEM 3.9

Solution. We shall take an element of the boiler with the width equal to the pitch a, of the rivets. The diagram for the calculation has the form shown in the figure. The stresses are determined as follows:

$$\sigma = \frac{qDa}{2(a-d)t} = \frac{4 \times 150 \times 7{\cdot}5}{2 \times (7{\cdot}5 - 2) \times 1} = 408 \text{ kg/cm}^2,$$

$$\tau = \frac{qDa}{2\,\frac{\pi d^2}{4}} = \frac{4 \times 150 \times 7{\cdot}5}{2 \times 3{\cdot}14} = 717 \text{ kg/cm}^2,$$

$$\sigma_c = \frac{qDa}{2td} = \frac{4 \times 150 \times 7{\cdot}5}{2 \times 1 \times 2} = 1125 \text{ kg/cm}^2.$$

The nearest to the allowable stresses were found to be the shear stresses. They determine the allowable pressure q. Since τ is directly proportional to q, the allowable pressure

$$q = 4 \times \frac{720}{717} = 4{\cdot}02 \text{ atm}.$$

3.10. A boiler 160 cm in diameter is subjected to an internal pressure of 10 atm. The longitudinal joint is designed as a lap joint with a double row of rivets. The thickness, t, of the plates is 10 mm, the diameter d, of the rivets is 20 mm and the pitch a, (the distance between the rivets of the same row) of the rivets is 100 mm. (1) Determine the shear stress, the bearing stress and the tensile stress in the joint. (2) Determine the limiting allowable pressure

in the boiler, if permissible stresses are: $[\tau] = 700\,\text{kg/cm}^2$, $[\sigma_c] = 1600\,\text{kg/cm}^2$, and $[\sigma] = 1000\,\text{kg/cm}^2$.

Answer (1) $\tau = 1273\,\text{kg/cm}^2$, $\sigma_c = 2000\,\text{kg/cm}^2$, $\sigma = 1000\,\text{kg/cm}^2$; (2) $q = 5{\cdot}5$ atm.

3.11. Determine what must be the diameter of the rivets in Problem 3.10 for the same pitch a, of 100 mm so that, utilizing the full strength of the rivets and plate, σ and τ may simultaneously be brought up to $[\sigma]$ and $[\tau]$ respectively. What may then be the allowable pressure in the boiler?

Answer: $d = 2{\cdot}6$ cm; $q = 9{\cdot}25$ atm.

3.12. The plates of a boiler, having a thickness of 20 mm are connected by a butt joint with two straps. In each half of the joint there are two staggered rows of rivets. The distance between the rows is 75 mm, the pitch in each row is 90 mm. The diameter of the rivets is 20 mm, the diameter of the boiler is 150 cm, and the thickness of the straps is 10 mm. Find the allowable internal pressure of the boiler, if the allowable stresses $[\tau]$, $[\sigma_c]$ and $[\sigma]$ are 700 kg/cm², 1600 kg/cm², and 1000 kg/cm² respectively.

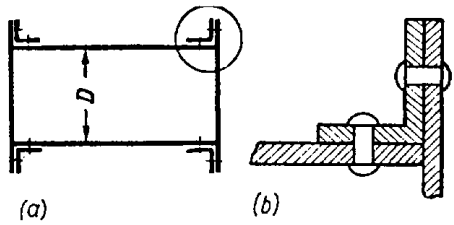
Answer: 13 atm.

3.13. The plates of a boiler are connected by a butt joint with two straps. In each half of the joint the rivets are arranged in two rows. The thickness of the plates is 20 mm; the thickness of the straps is 10 mm. The diameter of the boiler is 250 cm; the diameter of the rivets is 23 mm. The pitch of the rivets is 75 mm. The internal pressure in the boiler is 12 atm. Determine the tensile stress, the bearing stress and the shear stress in the joint.

Answer: $\sigma = 1080\,\text{kg/cm}^2$; $\sigma_c = 1223\,\text{kg/cm}^2$; $\tau = 676\,\text{kg/cm}^2$.

3.14. The end plates of a boiler are joined to the walls by rivets and angles as shown in the figure. The diameter of the boiler is 100 cm; the pressure inside the boiler is 10 atm. The thickness of the walls of the boiler and that of the legs of the angle are 10 mm. Determine the number of rivets required for joining the walls to the angle, if the diameter of the rivets is 20 mm. The allowable stresses are 700 kg/cm² in shear and 1600 kg/cm² in compression.

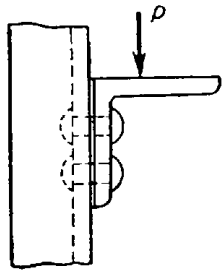
Answer: 36.



PROBLEM 3.14

3.15. A bracket is made of an angle, with dimensions of 150 mm × × 150 mm × 12 mm, fastened by five rivets, 20 mm in diameter, to a No.24a channel which forms a part of a column (see figure). Determine the shear stress and the bearing stress in the rivets. $P = 12$ t.

Answer: $\tau = 763$ kg/cm^2; $\sigma_c = 1500$ kg/cm^2.



PROBLEM 3.15

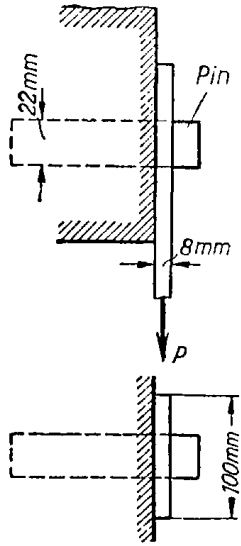
3.16. A copper bar 40 mm in diameter is inserted with a very small clearance into a steel tube with an outside diameter of 60 mm. At both ends the bar is fastened to the tube by rigid pins 20 mm in diameter, passing through the bar and both walls of the tube in a direction perpendicular to the axis of the bar. Determine the shear stresses in the pins if the temperature of the whole structure rises by 40°C. In determining thc forces the deformation of the pins should be not taken into account.

Answer: 229 kg/cm^2.

3.17. A pin 22 mm in diameter fastens a steel plate with a cross-section of 100 × 8 mm^2 to a wall (see figure). What are the tensile

stresses in the plate, and the bearing and shear stresses in the pin, if P is 4 t?

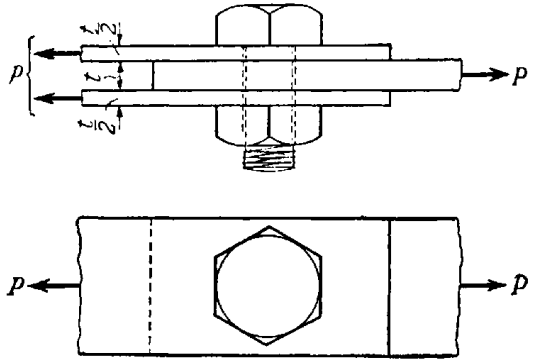
Answer: $\sigma = 642\,\text{kg/cm}^2$; $\sigma_c = 2270\,\text{kg/cm}^2$; $\tau = 1050\,\text{kg/cm}^2$.



PROBLEM 3.17

3.18. Determine the diameter of the bolt in the connection shown in the figure. The tensile force, P, is 20 t, the thickness, t, is 2 cm. The allowable stresses for the material of the bolt are 800 kg/cm² in shear and 2000 kg/cm² in compression.

Answer: 5 cm.



PROBLEM 3.18

3.19. The ultimate strength of mild steel in shear is 3800 kg/cm². Determine the force required to punch a hole: 1. (a) of 20 mm diameter in a plate 16 mm thick; (b) of 16 mm diameter in a plate 20 mm thick; (c) of 20 mm diameter in a plate 20 mm thick; (d) of 16 mm diameter in a plate 16 mm thick. 2. Find the corresponding bearing stress under the end of the punch.

Answer: 1. (a) 38·2t, (b) 38·2t, (c) 47·8t, (d) 30·6t; 2. 12,200 kg/cm², 19,000kg/cm²,15,200 kg/cm², 15,200 kg/cm².

3.20. Determine the maximum thickness of a steel sheet into which holes of 20 mm in diameter may be punched, if the ultimate strength of the steel in shear is 38 kg/mm², and the allowable bearing stress in the hardened end of the steel punch is 20,000 kg/cm².

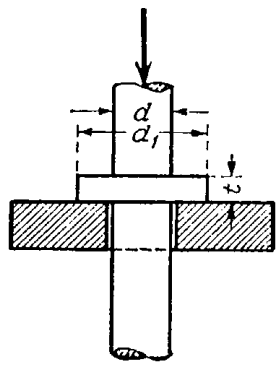
Answer: 26.4 mm.

3.21. Determine the minimum diameter d of a hole which may be punched into a sheet of a given thickness t, if the bearing stress in the punch is given as a quantity four times the ultimate strength of the material of the sheet in shear.

Answer: $d = t$.

3.22. The bearing stress under the collar of the bolt shown in the figure is 400 kg/cm², and the compressive stress in the bolt, 10 cm in diameter, is 1000 kg/cm². What is the diameter d_1 of the collar? Determine the shear stress in the collar if its thickness t is 5 cm.

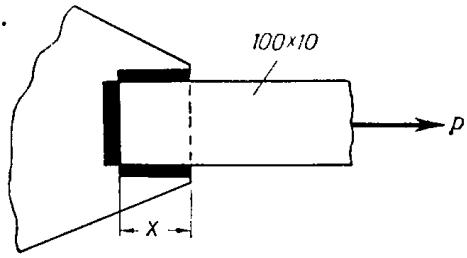
Answer: $d_1 = 18{\cdot}7$ cm; $\tau = 500$ kg/cm².



PROBLEM 3.22

3.23. Determine the minimum length, x, necessary for the weld on the steel plate shown in the figure, if the tensile stress in the plate is 1400 kg/cm², and the allowable shear stress $[\tau_c]$ for the weld is 800 kg/cm².

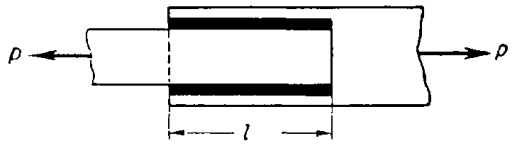
Answer: 7·5 cm.



PROBLEM 3.23

3.24. Determine the required length, l, of the flanking welds of a lap joint of two plates having different widths (see figure). The force, P, to which the joint is subjected is 15 t. The allowable shear stress for the weld is 1100 kg/cm². The thickness of the narrow plate is 10 mm, whereas that of the wide plate is 8 mm.

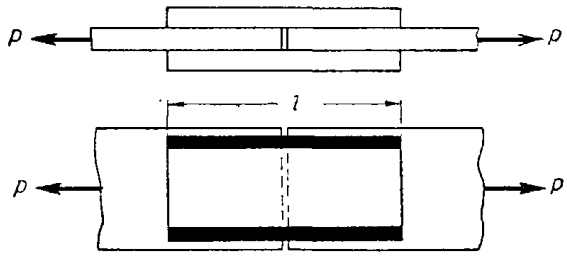
Answer: $l \approx 10$ cm.



PROBLEM 3.24

3.25. Determine the necessary length l of the cover plates of the butt joint (shown in the figure), pulled by a force P of 30 t. The thickness of the cover plates is 5 mm. The allowable stress of the weld in shear is 1100 kg/cm².

Answer: $l = 40$ cm.

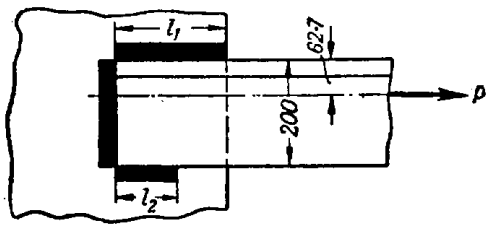


PROBLEM 3.25

3.26. Two plates, with the dimensions 250 mm × 10 mm, having two cover plates (with a cross-sectional area of 200 × 6 mm² each), are connected by a butt joint by means of flanking welds 40 cm long (see figure for Problem 3.25). To what tensile force may this joint be subjected, if the allowable stress for the plate and straps is 1600 kg/cm², and the allowable stress for the weld in shear is 1100 kg/cm²?

Answer: 38·4 t.

3.27. A tensile force of 40 t is applied centrally to a 150 mm × 200 mm × 16 mm angle. The angle is welded on to a plate as shown in the figure. To reduce the lengths l_1 and l_2 the end of the angle is also welded. It is required to determine the dimensions l_1 and l_2. The allowable stress for the weld in shear is 800 kg/cm².



PROBLEM 3.27

Solution. From the limiting condition of strength in shear we shall determine the total length of the seam

$$l = \frac{P}{0{\cdot}7t[\tau_e]} = \frac{40{,}000}{0{\cdot}7 \times 1{\cdot}6 \times 800} = 44{\cdot}7 \text{ cm}.$$

The length l_{end} of the end weld is 20 cm. The remaining length $l_{fl.}$ constitutes a total flanking seam of 24·7 cm.

The flanking seams l_1 and l_2 must be chosen in such a way that the resultant force, taken over the whole seam, would lie in the same plane as the axis of the angle—located at a distance of 6·27 cm from the back of the angle. Thus the following equations arise.

$$6{\cdot}27\, l_1 + \frac{6{\cdot}27^2}{2} = \frac{13{\cdot}73^2}{2} + 13{\cdot}73 l_2$$

and

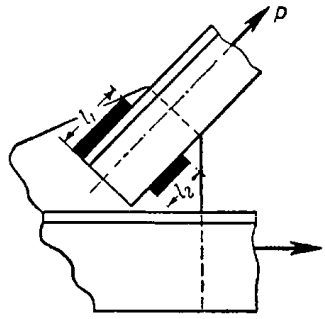
$$l_1 + l_2 = 24{\cdot}7.$$

Solving these simultaneous equations:

$$l_2 = 4 \text{ cm} \qquad \text{and} \qquad l_1 = 20{\cdot}7 \text{ cm}.$$

3.28. A tie consisting of an angle, with the dimensions 80 mm × × 80 mm × 8 mm, is fastened to a gusset plate of a joint by means of flanking welds as shown in the figure. It is subjected to a tensile force P of 6·2 t. Determine the necessary length of the weld on each side of the angle, if $[\tau_e]$ is 800 kg/cm².

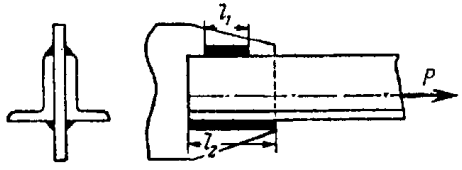
Answer: $l_2 = 3{\cdot}92$ cm; $l_1 = 9{\cdot}88$ cm.



PROBLEM 3.28

3.29. A tie consists of two angles, with the dimensions 120 mm × × 80 mm × 10 mm. They are welded on to a gusset plate by the flanking seams as shown in the figure. Determine the dimensions of the seams l_1 and l_2, if the allowable tensile stress in the angles is 1600 kg/cm², and the allowable stress for the welds in shear is 1100 kg/cm². The thickness of the seams is equal to the thickness of the angles.

Answer: $l_1 = 13$ cm; $l_2 = 26{\cdot}8$ cm.

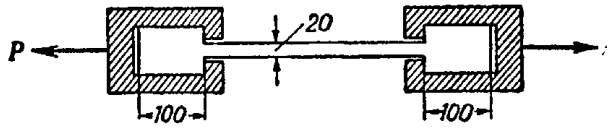


PROBLEM 3.29

3.30. What breaking force must be applied to a flat wooden test-piece with a cross-section of 2 × 4 cm² (see figure), if the ultimate

tensile strength for the wood is 560 kg/cm²? What is then the shear stress in the heads of the test-piece?

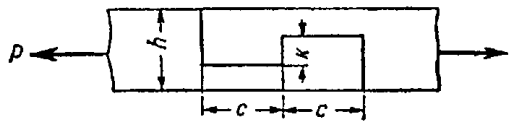
Answer: 4480 kg; 56 kg/cm².



PROBLEM 3.30

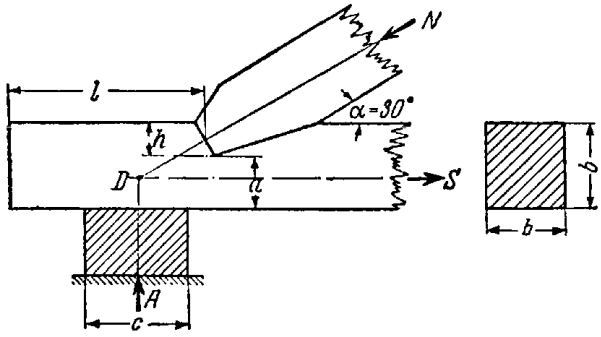
3.31. Determine the required dimensions of the joint shown in the figure. The tensile force P is 4 t. The bars are of square cross-section. The stresses are: $[\sigma] = 100$ kg/cm² in tension, $[\sigma_c] = 80$ kg/cm² in compression and $[\tau] = 10$ kg/cm² in shear.

Answer: $h = 11{\cdot}4$ cm; $k = 4{\cdot}4$ cm; $c = 35{\cdot}1$ cm.



PROBLEM 3.31

3.32. Determine the required depth h of the cut, the length l of the end of the tie bar, the width c of the support pad, and check the sufficiency of the height a at the weakest point of the support junction of a truss represented in the figure. The compressive force N in the rafter of the truss is 7 t. The width b of the bars is 20 cm. The allowable stresses are: $[\sigma] = 100$ kg/cm² in tension, $[\sigma_c]_{90} = 25$ kg/cm² in compression across the grain, $[\sigma_c]_{30} = 50$ kg/cm² in compression at an angle of 30° to the direction parallel to the



PROBLEM 3.32

grain, and $[\tau] = 12$ kg/cm² in shear along the grain. Assume that the force is not transmitted across the section n–t.

Solution. The forces N (in the rafter), S (in the tie bar) and the reaction A of the support at point D, are in equilibrium. Proceeding from this: $S = N \cos \alpha$, $A = N \sin \alpha$. The required dimensions are determined from the limiting conditions of strength:

(a) of the tie bar in compression due to force N

$$\frac{N}{b \dfrac{h}{\cos \alpha}} \leq [\sigma_c]_\alpha; \qquad h \geq \frac{N \cos \alpha}{b[\sigma_c]_\alpha} = \frac{7000 \times 0{\cdot}87}{20 \times 50} = 6{\cdot}1 \text{ cm};$$

(b) of the end of the tie bar in shear due to force S

$$\frac{S}{bl} \leq [\tau]; \qquad l \geq \frac{S}{b[\tau]} = \frac{N \cos \alpha}{b[\tau]} = \frac{7000 \times 0{\cdot}87}{20 \times 12} = 25{\cdot}2 \text{ cm};$$

(c) of the tie bar in compression due to force A

$$\frac{A}{bc} \leq [\sigma_c]_{90}; \qquad c \geq \frac{A}{b[\sigma_c]_{90}} = \frac{N \sin \alpha}{b[\sigma_c]_{90}} = \frac{7000 \times 0{\cdot}5}{20 \times 25} = 7 \text{ cm};$$

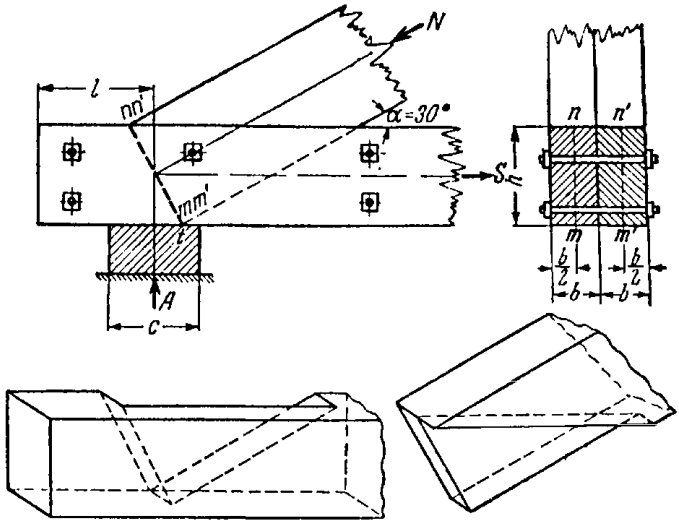
(d) of the tie bar in tension by force S

$$\frac{S}{ba} \leq [\sigma]; \qquad a \geq \frac{S}{b[\sigma]} = \frac{N \cos \alpha}{b[\sigma]} = \frac{7000 \times 0{\cdot}87}{20 \times 100} = 3{\cdot}04 \text{ cm}.$$

Due to constructional considerations, this dimension is considerably larger, and namely: $a = b - h = 20 - 6{\cdot}1 = 13{\cdot}9$ cm.

3.33. The joint at the support of a truss is designed in the form shown in the figure. The compressive force N in the rafter of the truss is 5 t. Check the strength of the joint in the plane $n\,n'\,m\,m'$, and determine also the required length l of the end of the tie bar, and the width c of the support pad. The width b of the planks is 6 cm; their height h is 20 cm. The allowable stresses are: $[\sigma_c]_{90} = 25$ kg/cm² in compression across the grain, $[\sigma_c]_{30} = 45$ kg/cm² in compression at an angle of 30° to the direction of the grain, and $[\tau] = 4{\cdot}8$ kg/cm² in shear along the grain. The bolts are not to be taken into account in the calculation. Assume that the force is not transmitted across the line n–t.

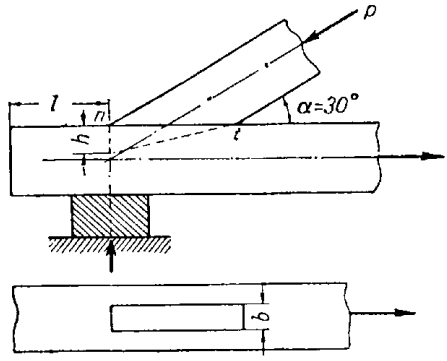
Answer: $\sigma_c = 36\,\text{kg/cm}^2 < 45\,\text{kg/cm}^2$; $l = 23\,\text{cm}$; $c = 8{\cdot}3\,\text{cm}$.



PROBLEM 3.33

3.34. At its support, the rafter of a truss is joined to the tie bar by means of a tenon with the width b 6 cm as shown in the figure. Determine the height h of the tenon (the depth of the cut), and the length l of the end of the tie bar, if the allowable stress in compression, at an angle of 30° to the direction of the grain, is 50 kg/cm², and the allowable stress in shear along the grain is 8 kg/cm². The compressive force P in the rafter of the truss is 2 t. Assume that the force is not transmitted across the line n–t.

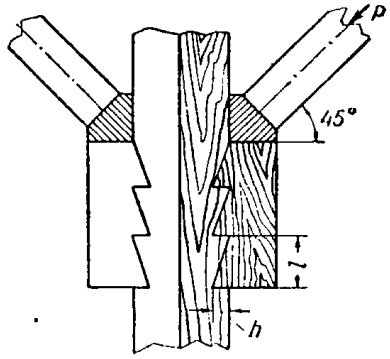
Answer: $h = 6$ cm; $l = 12$ cm.



PROBLEM 3.34

3.35. The knee brace of a bridge rests on an oak pad which by means of a toothed block transmits the force to the stanchion as shown in the figure. Determine the depth h of the cut, and the length l of the tooth, if the force P acting on the knee brace is 20 t. The cross-section of the knee brace and the stanchion is a square of 20×20 cm^2. The allowable stress is 80 kg/cm^2 in compression, and 8 kg/cm^2 in shear.

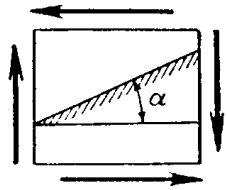
Answer: $h = 3$ cm; $l = 30$ cm.



PROBLEM 3.35

3.36. In an element subjected to pure shear with shear stresses of 500 kg/cm^2 find, by constructing Mohr's circle, the normal, shear and resultant stresses in a plane inclined at an angle α of 30° to the planes of shear as shown in the figure.

Answer: $\sigma_\alpha = 433$ kg/cm^2; $\tau_\alpha = 250$ kg/cm^2; $p_\alpha = 500$ kg/cm^2.



PROBLEM 3.36

3.37. A steel cube with an edge of 20 cm is subjected in its four faces to pure shear, with shearing stresses τ of 1000 kg/cm^2. Find the magnitude of the shear deformation and shear strain. Modulus of rigidity G is 8×10^5 kg/cm^2.

Answer: $\Delta s = 0{\cdot}25$ mm; $\gamma = 0{\cdot}00125$.

3.38. Determine the modulus of elasticity E for a material having a modulus of rigidity G of 4×10^5 kg/cm^2 and Poisson's ratio μ of 0·2.

Answer: $E = 0{\cdot}96 \times 10^6$ kg/cm^2.

3.39. Before a flat steel test-piece was loaded a line was drawn on it at an angle of 30° to its axis. By what amount will this angle change, if the test-piece is stretched until the stresses, σ, are 1200 kg/cm^2? $E = 2 \times 10^6$ kg/cm^2, $\mu = 0{\cdot}25$.

Answer: $\gamma = 0{\cdot}00065$ radians $= 2'14''$.

11. TORSION OF ROUND BARS

3.40. A hollow steel shaft has an external diameter d_1 of 100 mm, and internal diameter d_2 of 50 mm. What horsepower does this shaft transmit rotating at a speed n of 80 rev/min, if it is twisted through an angle of 1·8° over a length of 2·7 m? What is the maximum shear stress in the shaft?

Solution. The torque carried by the shaft can be determined from the formula $\varphi = M_k l/J_p G$:

$$M_k = \frac{\varphi J_p G}{l}. \tag{a}$$

The power N is related to the torque as follows

$$M_k = \frac{2250}{\pi} \times \frac{N}{n} \text{ kgm} = \frac{225{,}000}{\pi} \times \frac{N}{n} \text{ kgcm}. \tag{b}$$

Hence, after substituting for M its value from equation (a)

$$N = \frac{\pi n \varphi J_p G}{225{,}000\, l}.$$

We express the angle φ in radians:

$$\varphi = 1{\cdot}8° = 1{\cdot}8 \frac{\pi}{180} = \frac{\pi}{100}.$$

The polar second moment of area

$$J_p = \frac{\pi r_1^4}{2}(1 - \alpha^4) = \frac{15}{32}\pi r_1^4.$$

Then

$$N = \frac{\pi \times 80 \times \pi \times 15 \times \pi \times 5^4 \times 8 \times 10^5}{100 \times 32 \times 22500 \times 270} = 95{\cdot}6 \text{ h.p.}$$

The maximum shear stress is expressed by the formula

$$\max \tau = \frac{M_k r_1}{J_p}. \tag{c}$$

Substituting for M_k its value expressed in terms of the power N and the given rotational speed we obtain

$$\max \tau = \frac{225{,}000 N r_1}{\pi n J_p} = \frac{225{,}000 \times 95{\cdot}6 \times 32 \times 5}{\pi \times 80 \times 15 \times \pi \times 5^4} = 466 \text{ kg/cm}^2.$$

3.41. Determine the diameter of a solid shaft transmitting a torque of 1·5 tm, if the allowable stress $[\tau]$ is 700 kg/cm².

Answer: 10·3 cm.

3.42. Determine the diameter of a solid shaft transmitting 450 h.p. at 300 rev/min. The angle of twist must not exceed 1° over a 2 m length of the shaft, and the maximum shear stress is 400 kg/cm². The modulus of rigidity G is 8×10^5 kg/cm².

Answer: 11·2 cm.

3.43. Determine the smallest diameter of a steel shaft transmitting 18 h.p. at 120 rev/min, if the allowable angle of twist is 1° over a length equal to 15 diameters of the shaft. What will then be the maximum shear stresses?

Answer: 4·9 cm; 466 kg/cm².

3.44. Compare the weights of two solid shafts of identical length: one of steel, and the other of an aluminium alloy—both designed for the same angle of twist per unit length when subjected to equal torques. $G_a = 2{\cdot}7 \times 10^5$ kg/cm², $G_s = 8 \times 10^5$ kg/cm²; $\gamma_a = 2{\cdot}6$; $\gamma_s = 7{\cdot}85$.

Answer: $Q_s = 1{\cdot}75\ Q_a$.

3.45. To determine the required diameter of a solid steel shaft an allowable shear stress of 800 kg/cm², and allowable angle of twist of 1° over a length of 1 m are given. For what values of torque may the choice of the diameter be limited only by the allowable stress, or only by the allowable angle of twist?

Answer: For $M_k > 2{\cdot}36$ t m by the allowable stress;
for $M_k < 2{\cdot}36$ t m by the allowable angle of twist.

3.46. A solid shaft with a diameter of 10 cm and length of 6 m is twisted through an angle of 4°. What is the maximum shear stress if G is 8×10^5 kg/cm²?

Answer: 466 kg/cm².

3.47. A solid shaft 90 mm in diameter transmits 50 h.p. at a rotational speed of 150 rev/min. The length of the shaft between the pulleys is 4 m. The modulus G is 8×10^5 kg/cm². Determine the maximum shear stress in the shaft, and the angle through which one pulley is turned with respect to the other.

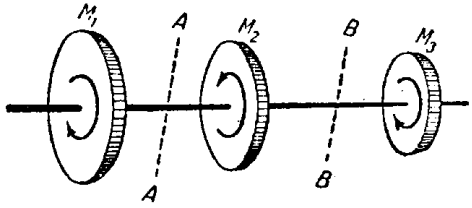
Answer: 167 kg/cm²; 1°4′.

3.48. An approximate formula for determination of the diameter of a shaft has the form $d = \sqrt[3]{2{\cdot}5(N/n)}$, where d is the diameter of the shaft, N is the power being transmitted in h.p., and n is the number of revolutions per minute. On what value of the allowable shear stress is this formula based?

Answer: 600 kg/cm².

3.49. Determine the maximum shear stresses in sections AA and BB of the shaft (see figure) if M_1 is 13 t cm and M_2 is 30 t cm. The diameter of the shaft at section AA is 5 cm, whereas at section BB it is 7·5 cm.

Answer: $\tau_A = 530$ kg/cm²; $\tau_B = 205$ kg/cm².



PROBLEM 3.49

3.50. A steel testpiece 20 mm in diameter having an effective length of 200 mm is tested in torsion. The results of the test are given in the table:

Torque, kgm	2	4	6	8	10	12	14	16	18
Angle of twist in radians × 2000	6·1	12·5	18·9	25·2	31·4	37·8	44·3	50·5	56·9

Torque, kgm	20	22	24	26	28	30	32	34
Angle of twist in radians × 2000	63·0	69·5	75·8	82·1	88·5	95·4	110	148

Construct a diagram of the shear stresses versus the angle of twist, and determine the modulus of rigidity. Find the limit of proportionality for the case under consideration.

3.51. Determine the maximum torque which may be applied to a steel rod 10 mm in diameter, if the allowable stress is not to exceed 1500 kg/cm². What is the minimum length of the rod, if the angle of twist is 90°? The modulus G is 8×10^5 kg/cm².

Answer: 295 kgcm; 418 cm.

3.52. A shaft 90 mm in diameter transmits 90 h.p. Determine the limiting rotational speed, if the allowable shear stress is 600 kg/cm².

Answer: $n \geq 75$ rev/min.

3.53. A steel shaft 2 m long and 5 cm in diameter is twisted through an angle of 9·2° when loaded by a torque of 400 kgm. The limit of proportionality for the shear stresses is 1700 kg/cm². Determine the value of the modulus of rigidity.

Answer: $8{\cdot}1 \times 10^5$ kg/cm².

3.54. A mild steel bar 25 mm in diameter is elongated by 0·113 mm over a length of 20 cm when subjected to a force of 6 t. The same bar is twisted through an angle of 0·55° over a length of 15 cm

when loaded by a torque of 2000 kg cm. Determine the value of E, G and μ.

Answer: $2{\cdot}16 \times 10^6$ kg/cm^2; $8{\cdot}16 \times 10^5$ kg/cm^2; 0·32.

3.55. While testing (in torsion) a steel test-piece, 20 cm long and 20 mm in diameter, it was found that at a torque of 1640 kg cm the angle of twist was 0·026 radians. The limit of proportionality was reached when the torque was 2700 kg cm. Determine the value of the modulus of rigidity, and the limit of proportionality in torsion.

Answer: $8{\cdot}0 \times 10^5$ kg/cm^2; 1720 kg/cm^2.

3.56. A hollow shaft connecting a turbine and generator in a hydro-electric installation has an external diameter of 40 cm and internal diameter of 22·5 cm. The speed of rotation is 120 rev/min. What are the maximum shear stresses if the shaft transmits 10,000 h.p.?

Answer: 530 kg/cm^2.

3.57. While determining the power of a steam turbine the angle of twist of the rotating shaft was measured, and was found to be 1·2° over a length of 6 m. The external and internal diameters of the shaft are 25 cm and 17 cm respectively. The rotational speed of the shaft is 250 rev/min. The modulus G is 8×10^5 kg/cm^2. Determine the power being transmitted by the shaft, and the shear stresses set up in it.

Answer: 2940 h.p.; 349 kg/cm^2.

3.58. To determine the power being transmitted by a shaft its elongation was measured by means of an extensometer along a line inclined to the external generator of the shaft at an angle of 45°. The measured strain was found to be 0·000425. The external diameter of the shaft is 40 cm, and the internal diameter is 24 cm. The modulus of rigidity G is 8×10^5 kg/cm^2. What is the power being transmitted by the shaft, if it rotates at a speed of 120 rev/min? What are then the maximum shear stresses?

Answer: 12,500 h.p.; 680 kg/cm^2.

3.59. A solid shaft 40 cm in diameter is replaced by a hollow shaft in which the internal diameter constitutes 60% of the external diameter. Determine the external diameter of the hollow shaft,

with the condition that the allowable shear stresses in both are equal. Compare the weight of the solid shaft with that of the hollow shaft.

Answer: 42 cm; $Q_{\text{solid}}/Q_{\text{hollow}} = 1{\cdot}41$.

3.60. In order to reduce the weight of a solid shaft by 20% it is decided to replace it with a hollow shaft, the external diameter of which is twice its internal diameter. What will be the maximum shear stresses in the hollow shaft, if in the solid shaft they were 600 kg/cm²?

Answer: 580 kg/cm².

3.61. Two shafts, one of which is solid and the other hollow, have the same weight, and transmit equal torques. Calculate the ratio of the maximum shear stress in one shaft to that in the other, stating in which shaft the greatest stress is encountered. The internal diameter of the hollow shaft is 0·6 times the external diameter.

Answer: In the hollow shaft; 1·7 times.

3.62. Determine the external diameter of a hollow steel shaft transmitting 9600 h.p. at 110 rev/min, if the allowable shear stress is 560 kg/cm², and the inside diameter is 0·6 times the outside diameter.

Answer: 40·2 cm.

3.63. A hollow steel shaft with a length of 1·8 m is loaded by a torque of 0·6 tm. Determine the external and internal diameters of the shaft, if the angle of twist is not to exceed 2°, and the shear stress is not to exceed 700 kg/cm².

Answer: 90·4 mm; 72·4 mm.

3.64. Plot a curve, the abscissa of which is the rotational speed in revolutions per minute, and the ordinate the required diameter of a solid steel shaft transmitting 50 h.p. with an allowable shear stress of 600 kg/cm². Vary the speed within the limits of 16–16,000 rev/min.

Answer: $d = 31{\cdot}2 / \sqrt[3]{n}$.

3.65. A torque M_k of 62·5 tcm is applied to a thin-walled pipe with a mean diameter d equal to 12·5 cm. What must be the wall thickness t of the pipe, in order that the shear stresses $[\tau]$ would

not exceed 800 kg/cm²? Find the angle of twist over a length l of 1 m, if G is 8×10^5 kg/cm².

Solution. In torsion of thin-walled bars the shear stresses at the inner and outer fibres differ so little from each other that they are taken as equal to the mean. Then they can be determined by the formula $\tau = M_k r_m / J_p$, where r_m is the mean radius of the annular cross-section. Because of the small difference between the external and internal radii of the cross-section its polar second moment of area may be calculated as follows

$$J_p = \int_F dF \varrho^2 = 2\pi r_m t r_m^2 = 2\pi r_m^3 t,$$

and the formula for the determination of the stresses has the form

$$\tau = \frac{M_k}{2\pi r_m^2 t}.$$

The wall thickness t is determined from the equation

$$t = \frac{M_k}{2\pi r_m^2 [\tau]} = \frac{62{,}500}{2 \times \pi \times 6{\cdot}25^2 \times 800} = \frac{1}{\pi} = 0{\cdot}318 \text{ cm}.$$

The angle of twist is found by the usual formula

$$\varphi = \frac{M_k l}{J_p G} = \frac{62{,}500 \times 100\pi}{2 \times \pi \times 6{\cdot}25^3 \times 1 \times 8 \times 10^5} = 0{\cdot}016 \text{ radians}.$$

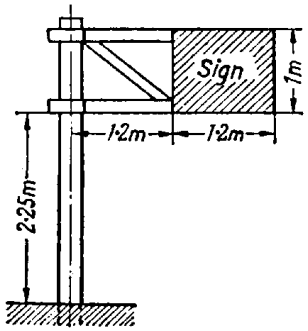
3.66. A tubular shaft must transmit 100 h.p. whilst rotating at a speed of 100 rev/min. The wall thickness constitutes 1/50 of the mean diameter of the tube. Determine the mean diameter of the tube, with the condition that the angle of twist over a length of 3 m must not exceed 1°. The modulus G is 8×10^5 kg/cm². What will then be the shear stresses in the transverse section of the tube?

Answer: 17·7 cm; 412 kg/cm².

3.67. A steel tube is to be used as a post for a road sign, as indicated in the figure. The maximum wind pressure on the sign is assumed to equal 200 kg/m². The angle of rotation of the tube at the level of the lower strap of the sign must not exceed 6°. The

maximum shear stresses, due to torsion in the transverse section of the tube must not be greater than 350 kg/cm². Determine the mean diameter of the tube, if its wall thickness is 3 mm. Assume that the wind pressure is transmitted only over the area shaded in the figure.

Answer: 16·2 cm.



PROBLEM 3.67

3.68. A pipe with a length of 4·6 m and mean diameter of 15 cm has a wall thickness of 2·5 mm. The shear stresses in it are 560 kg/cm². Find the total angle of twist of the tube.

Answer: 2·46°.

3.69. A shaft of tubular cross-section 1·8 m long, with a mean diameter of 30 cm and wall thickness of 3 mm, rotates at a speed of 100 rev/min. What horsepower does it transmit, if the shear stress is 630 kg/cm²? Determine the angle of twist.

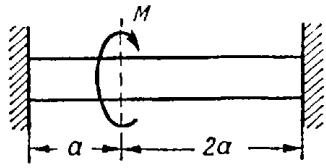
Answer: 374 h.p.; 0·542°.

3.70. A tubular shaft with a mean diameter of 10 cm has a wall thickness of 3 mm. The angle of twist must not exceed 1° over a length of 3·7 m. Determine the rotational speed of the shaft at which it can transmit 50 h.p.

Answer: 404 rev/min.

3.71. The ends of a round bar are firmly clamped as shown in the figure. At an intermediate section of the rod a torque of 1200 kgm is applied. Determine the maximum shear stresses, if the diameter of the shaft is 8 cm.

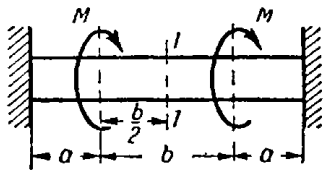
Answer: 795 kg/cm².



PROBLEM 3.71

3.72. A round bar with the ends clamped rigidly is subjected to two equal torques 800 kgm each, acting in the same direction (see figure). Determine the angle of rotation of the middle section 1–1, if the diameter of the shaft is 10 cm. The distances a and b are 60 cm and 80 cm respectively.

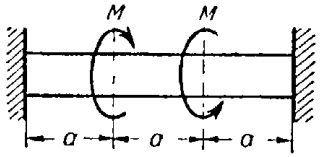
Answer: 21′.



PROBLEM 3.72

3.73. Two equal and opposing torques 1000 kgm each (see figure) are applied to a round bar having rigidly clamped ends. Determine the diameter of the shaft, if the allowable shear stress is 600 kg/cm².

Answer: 8·25 cm.



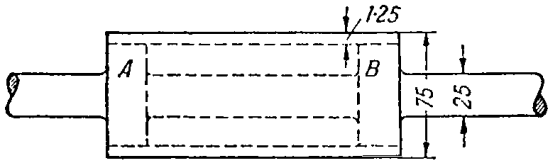
PROBLEM 3.73

3.74. A copper tube with an external diameter of 7·5 cm is placed inside a steel tube of the same internal diameter. The wall thickness of both tubes is 3 mm each. The ends of the tubes are rigidly fixed to each other, and a torque of 100 kgm is applied to them. Determine the maximum shear stresses in each tube, the distri-

bution of the torque between the tubes, and the angle of twist over a length of 3 m. $G_c = 4 \times 10^5$ kg/cm^4; $G_s = 8 \times 10^5$ kg/cm^2.

Answer: 250 kg/cm^2; 115 kg/cm^2; 71·8 kgm; 28·2 kgm; 1·38°.

3.75. A steel bar 2·5 cm in diameter has two flanges A and B (see figure). A steel tube with a wall thickness of 1·25 mm is pulled on to these flanges. While the tube was fitted the bar was subjected to a torque of 7·5 kgm. After the tube was welded to the flanges the torque was removed from the ends of the bar. Determine the magnitude of the shear stresses in the tube after the removal of the torque from the ends of the bar. Assume that the flanges A and B do not deform.



PROBLEM 3.75

Solution. After the removal of the couples M_0 from the ends of the bar the part of it between the flanges A and B, which has been twisted through an angle φ_0, tries to untwist but is restrained by the tube. Therefore the bar is not completely untwisted, but only through an angle φ_t—through which the tube is now twisted. Thereby a couple M_t is transmitted from the bar, via the flanges A and B, to the tube. An equal couple M_b is also transmitted from the tube to the bar. Thus, for the solution of the problem it is possible to set up the two equations:

the equation of equilibrium

$$M_t = M_b = M_k,$$

and the equation of compatibility of deformations

$$\varphi_0 - \varphi_t = \varphi_b,$$

or

$$\frac{M_0 l}{J_{p(b)} G} - \frac{M_t l}{J_{p(t)} G} = \frac{M_b l}{J_{p(b)} G}.$$

After eliminating l and G, and replacing M_t and M_b by the quantity M_k we have

$$\frac{M_0}{J_{p(b)}} = \frac{M_k}{J_{p(t)}} + \frac{M_k}{J_{p(b)}} = \frac{M_k}{J_{p(b)}}\left(1 + \frac{J_{p(b)}}{J_{p(t)}}\right),$$

whence

$$M_k = \frac{M_0}{1 + \dfrac{J_{p(b)}}{J_{p(t)}}} = \frac{7{\cdot}5}{1 + 0{\cdot}0975} = 6{\cdot}85 \text{ kg m}.$$

Now it will be possible to determine the shear stresses in the tube

$$\tau = \frac{M_k}{2\pi r_m t} = \frac{685}{2 \times 3{\cdot}14 \times 3{\cdot}69^2 \times 0{\cdot}125} = 64{\cdot}3 \text{ kg/cm}^2.$$

12. TORSION OF BARS WITH NON-CIRCULAR CROSS-SECTION. JOINTS WORKING IN TORSION

3.76. A steel rod 2 m long, of rectangular cross-section with the dimensions 10 mm × 30 mm, is loaded by a torque of 1000 kg cm. Determine (a) the position and magnitude of the maximum shear stresses; (b) the magnitude of the shear stresses at the middle of the short side of the cross-section; (c) the angle of twist of the rod.

Answer: (a) 1250 kg/cm²; (b) 940 kg/cm²; (c) 18·2°.

Instruction. The angle of twist and the maximum shear stresses for rods of rectangular cross-section subjected to torsion are calculated by the formulae coinciding with the formulae for round rods, and namely:

$$\varphi = \frac{M_k l}{J_k G}; \qquad \tau_{max} = \frac{M_k}{W_k}.$$

In these formulae J_k and W_k respectively are quantities of the same dimensions as the second moment of area and section modulus. They are determined by the formulae: $J_k = \alpha b^4$, $W_k = \beta b^3$. Here b is the dimension of the smaller side of the cross-section. The maximum shear stresses occur in the mid-point of the long side of the cross-section. The shear stresses in the middle of the

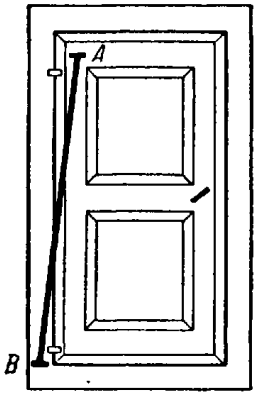
short side of a cross-section are determined by the formula $\tau = \gamma \tau_{max}$. The coefficients α, β, and γ depend on the ratio h/b of the sides, and are given in the table:

Values of α, β, and γ for rectangular cross-sections

h/b	1	1·5	2·0	3·0	4·0	6·0	8·0	10·0
α	0·140	0·294	0·457	0·790	1·123	1·789	2·456	3·123
β	0·208	0·346	0·493	0·801	1·150	1·789	2·456	3·123
γ	1·0	0·859	0·795	0·753	0·745	0·743	0·742	0·742

3.77. A door spring in the form of a steel rod of square cross-section is fastened by one end to the door (at point A in the figure), and by the other end to the lintel (at point B in the figure). Find the force P with which it will be necessary to pull the handle to open the door through 90°, and the angle α through which it will be necessary to twist the rod beforehand, so that during the opening of the door the maximum stresses in the rod would be 5000 kg/cm². The length of the rod is 2 m, the dimensions of the cross-section are 6 mm × 6 mm. The width of the door is 1 m.

Answer: 2·25 kg; 87°.



PROBLEM 3.77

3.78. Determine the dimensions of a shaft of square cross-section transmitting 100 h.p. at 120 rev/min. The shear stresses must not exceed 450 kg/cm^2.

Answer: 86 $\times$ 86 mm^2.

3.79. The rectangular cross-section of a rod has a width of 4 cm. What should be the depth of the cross-section for the maximum shear stresses in it to be the same as in a rod of circular cross-section 5 cm in diameter, if both rods are loaded by equal torques.

Answer: 6·52 cm.

3.80. A rod of rectangular cross-section is loaded by a torque of 60 kgm. What must be the depth of the cross-section if its width is 20 mm? The rod is made of steel with a limit of proportionality of 3500 kg/cm^2 in tension. The limit of proportionality in shear is 0·6 times the limit of proportionality in tension, and the allowable shear stress is assumed to be 2/3 of the limit of proportionality.

Answer: 4·28 cm.

3.81. A portion of a circular shaft was machined to fit a pulley—and was given the cross-section of a square inscribed in the circle. Determine how many times the maximum shear stresses in the square cross-section will increase in comparison with the maximum shear stresses in the circular cross-section of the shaft.

Answer: 2·66 times.

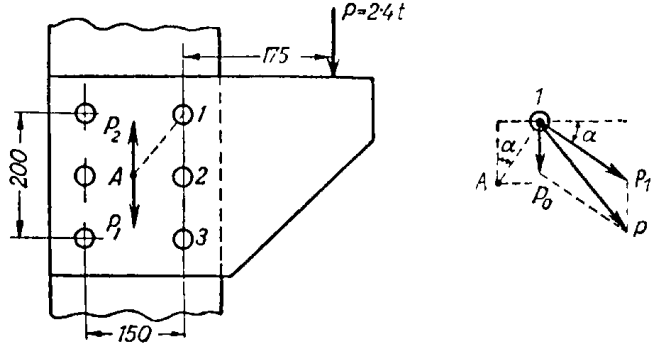
3.82. Two rods are loaded by equal torques: one of them is of circular cross-section, and the other is of square cross-section. Establish (a) in which rod the stresses are greater (and how many times greater) if their cross-sectional areas are equal, (b) what is the ratio of their weights, if the maximum stresses in both rods are equal.

Answer: (a) in the rod of square cross-section the stresses are 1·35 times greater; (b) the weight of the rod of square cross-section is 1·23 times greater.

3.83. Which rod is more suited to work in torsion: one of square cross-section, or one of rectangular cross-section (other conditions being equal), if the ratio of the sides of the rectangle is 0·5?

Answer: The weight of the rod of rectangular cross-section is 1·12 times greater.

3.84. Determine the resultant force on the most severely loaded rivet in the connection represented in the figure.



PROBLEM 3.84

Solution. The force P must be referred to the centroid of the rivet configuration, i.e. to the point A. Applying to this point two opposing forces $P_1 = P_2 = P$ we reduce the action of force P to the equivalent loading: the force P applied to the centroid of the connection, and the couple M_k equal to Pe, where e is the distance from the force to point A. The force P is divided evenly between the rivets, and the force on each of the rivets is defined as P/n, where n is the number of rivets.

The couple sets up torsion in the riveted connection. Thereby additional forces are set up in all rivets. It is assumed that these forces in the rivets are directly proportional to the distances of the rivets from the centroid (point A), and directed perpendicularly to the corresponding radii. Consequently, if we denote the force in any rivet (which is taken as a reference) by P_0, and the distance to it by ϱ_0, then the force P_i in anyother rivet located at a distance ϱ_i will be

$$P_i = \frac{P_0 \varrho_i}{\varrho_0}. \tag{a}$$

The sum of the moments of the forces in all rivets is equal to the applied couple:

$$M_k = Pe = \sum_1^n P_i \varrho_i.$$

After substituting for P_i the value from equation (a) we have

$$M_k = Pe = \frac{P_0}{\varrho_0} \sum_1^n \varrho_i^2, \qquad \text{or} \qquad \frac{P_0}{\varrho_0} = \frac{M_k}{\sum_1^n \varrho_i^2}.$$

Then the force in any rivet

$$P_i = \frac{M_k}{\sum_1^n \varrho_i^2} \varrho_i.$$

The total force acting on a rivet is the vector sum of the forces due to the force P, applied to the centroid, and the couple. The rivet with the maximum resultant force will be the most critical one. And this is the one to be checked for strength.

In this case point A lies at the distance e, equal to 250 mm, from the force. The distance to rivet No.1, $\varrho_1 = \sqrt{7{\cdot}5^2 + 10^2} = 12{\cdot}5$ cm. The distance to rivet No.2, $\varrho_2 = 7{\cdot}5$ cm. We calculate the quantity

$$\sum_1^n \varrho_i^2 = (4 \times 12{\cdot}5^2) + (2 \times 7{\cdot}5^2) = 625 + 112{\cdot}5 = 737{\cdot}5 \text{ cm}^2.$$

The force in rivet No.1,

$$P_1 = \frac{Pe\varrho_1}{\sum_1^n \varrho_i^2} = \frac{2{\cdot}4 \times 25 \times 12{\cdot}5}{737{\cdot}5} = 1{\cdot}018 \text{ t}.$$

The force in rivet No.2,

$$P_2 = P_1 \frac{7{\cdot}5}{12{\cdot}5} = 0{\cdot}610 \text{ t}.$$

The force on every rivet due to the centroidal force,

$$P_0 = \frac{P}{n} = \frac{2{\cdot}4}{6} = 0{\cdot}4 \text{ t}.$$

For rivet No.2 the force due to the couple is directed vertically downwards. Therefore the total force in it is $0{\cdot}4 + 0{\cdot}610 = 1{\cdot}01$ t.

We shall now determine the total force on rivet No.1. The force P_1 set up by the couple is directed perpendicularly to the radius, and acts at an angle $\alpha = \text{arc tan } 0{\cdot}75 = 36° 50'$ in relation to the horizontal.

The total force is determined as the vector sum of P_1 and P_0:

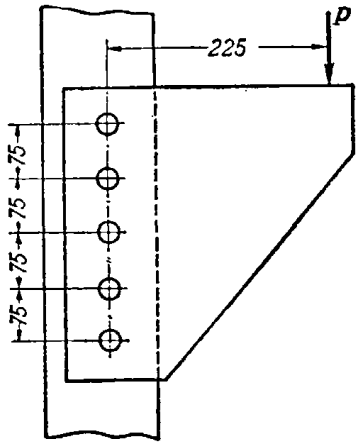
$$P = \sqrt{P_0^2 + P_1^2 + 2P_0P_1 \cos (P_0, P_1)} = \sqrt{P_0^2 + P_1^2 + 2P_0P_1 \sin \alpha}$$

$$= \sqrt{0{\cdot}4^2 + 1{\cdot}018^2 + (2 \times 0{\cdot}4 \times 1{\cdot}018 \times 0{\cdot}6)} = 1{\cdot}2 \text{ t}.$$

An equal force is carried by rivet No.3. Therefore, this is the force for which the strength of the rivet must be checked.

3.85. The riveted connection shown in the figure carries a force, P, of 3·5 t. The diameter of the rivets is 20 mm. Determine the shear stress in the critical rivet. The rivets are in single shear.

Answer: 704 kg/cm².

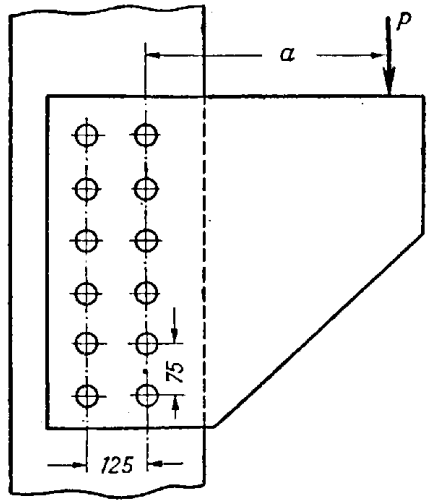


PROBLEM 3.85

3.86. In the connection shown in the figure the diameter of the rivets is 23 mm. The distance between the rows is 125 mm, whereas the pitch of the rivets is 75 mm. Determine the distance a, from the end row of the rivets to a force P of 1·6 t, if the allowable stress for the rivets in shear is 900 kg/cm². The thickness of the

plates is such that the bearing stresses do not exceed the allowable value. The rivets are in single shear.

Answer: $a \leq 116$ cm.



PROBLEM 3.86

3.87. Two parts of a shaft 10 cm in diameter are joined by means of bolts and flanges. The bolts are located on a circle 20 cm in diameter. Determine the required number of bolts 20 mm in diameter, if the allowable shear stress for the bolts is 600 kg/cm^2, and the maximum shear stress for the material of the shaft is 700 kg/cm^2.

Instruction. The torque carried by the shaft is transmitted to the bolted connection. Since all the bolts are located at the same distance from the centre of the connection, the forces in them will be equal. They are determined from the condition that $M_k = PR_0n$. M_k is determined from the limiting strength condition of the shaft, and P from the limiting strength condition of bolts.

Answer: 8 bolts.

3.88. Two shafts A and B, having diameters of 75 mm and 100 mm respectively, are joined by means of flanges with six bolts. The bolt diameter is 20 mm, and the bolt centres are located on a circle 25 cm in diameter. A torque of 600 kgm is transmitted through

the connection. Determine the maximum shear stresses in shaft A, and the shear stresses in the bolts.

Answer: 725 kg/cm^2; 254 kg/cm^2.

3.89. Two pipes are joined by means of flanges with 6 bolts 38 mm in diameter. The bolts are located at a distance of 18 cm from the axis of the pipe. Determine the magnitude of the torque being transmitted by the connection, if the shear stresses in the bolts are 560 kg/cm^2.

Answer: 3·43 t m.

3.90. Two shafts are joined by flanges. Four bolts 25 mm in diameter are arranged symmetrically on a circle 7·5 cm in radius. It was found that one bolt fits so loosely that it will not carry a load. Determine which one of the remaining bolts is most heavily loaded, and what are the shear stresses in the cross-section of this bolt.

Answer: The two symmetrically placed bolts will be loaded most; 269 kg/cm^2.

Instruction. It will be necessary to find the centroid of the remaining three bolts. The forces in the bolts will be proportional to their distances from the centroid. The moment of the forces about this point is equal to the applied torque; thus the resultant is equal to zero.

3.91. A pulley rotating with a speed of 200 rev/min transmits 125 h.p. to a solid steel shaft 75 mm in diameter. The pulley is secured to the shaft by means of a steel key of 1×1 cm^2 in cross-section and 10 cm in length. Determine the shear stresses in the key.

Answer: 1190 kg/cm^2.

13. HELICAL SPRINGS

3.92. A helical spring of circular cross-section wire 18 mm in diameter is loaded by a force, P, of 50 kg. The mean coil diameter D, of the spring is 125 mm. The modulus of rigidity G, is 8×10^5 kg/cm^2. Determine the maximum shear stresses in the

material of the spring. What number of coils must the spring have for its deflection to be 6 mm?

Solution. The maximum stress

$$\max \tau = \frac{P}{F} + \frac{M_k}{W_p} = \frac{P}{\pi r^2} + \frac{2PR}{\pi r^3}.$$

After insertion of the numerical values we have

$$\max \tau = \frac{50}{\pi \times 0{\cdot}9^2} + \frac{2 \times 50 \times 6{\cdot}25}{\pi \times 0{\cdot}9^3} = 19{\cdot}6 + 273 \approx 293 \text{ kg/cm}^2.$$

The deflection of the spring

$$\lambda = \frac{4PR^3 n}{Gr^4},$$

whence

$$n = \frac{\lambda G r^4}{4 \times P \times R^3} = \frac{0{\cdot}6 \times 8 \times 10^5 \times 0{\cdot}9^4}{4 \times 50 \times 6{\cdot}25^3} = 6{\cdot}46.$$

3.93. A helical spring made of 6 mm wire has 20 coils with a mean radius of 7·5 cm. Determine the axial tensile load which may be applied to the spring, if the shear stresses in it must not exceed 1800 kg/cm². What will then be the elongation of the spring and the maximum specific strain energy? The modulus G is 8×10^5 kg/cm².

Answer: 10 kg; 52·1 cm; 1·01 kg cm/cm³.

3.94. A wire 6 mm in diameter is to be wound into a helical spring, with a diameter of the generating cylinder equal to 5 cm*. This spring is to deflect 2·5 cm under a load of 9 kg. Determine the required length of wire. What will then be the angle of twist between the ends of the wire, and the maximum shear stress in the spring? The modulus G is 8×10^5 kg/cm².

Answer: 361 cm; 51°; 627 kg/cm².

* A cylinder on to which the wire is wound during the manufacture of a spring is called the generating cylinder.

3.95. A helical spring of steel wire 6 mm in diameter has 5 coils. The external diameter of the spring is 3·3 cm. Determine the stiffness of the spring, i.e. the force necessary to extend it by 1 cm, assuming that the limit of proportionality of the material is not to be exceeded. What value may be allowed for the deflection (elongation) of the spring, if the allowable stress for the material of the spring is 3000 kg/cm^2.

Answer: 132 kg/cm; 0·64 cm.

3.96. It is required to manufacture a helical spring which would deflect 2 cm under a load of 0·6 t. Determine the wire diameter if the spring is to be wound on a generating cylinder 20 cm in diameter, if the allowable shear stress is 750 kg/cm^2. What is the required number of coils, if the modulus G is 8×10^5 kg/cm^2?

Solution. We shall write down the equation of limiting stress

$$\frac{P}{\pi r^2}\left[1 + \frac{2(R_0 + r)}{r}\right] \leq [\tau]$$

and re-arrange it as follows

$$P + \frac{2PR_0}{r} + 2P = [\tau]\pi r^2,$$

$$r^2 = \frac{2PR_0}{\pi[\tau]r} + \frac{3P}{\pi[\tau]}.$$

After substituting the numerical values

$$r^2 = \frac{5{\cdot}08}{r} + 0{\cdot}764.$$

This equation is conveniently solved by trial values:

at $r = 2$ cm	$4 > 2{\cdot}54 + 0{\cdot}76 = 3{\cdot}3$;
at $r = 1{\cdot}8$ cm	$3{\cdot}24 < 2{\cdot}83 + 0{\cdot}76 = 3{\cdot}59$;
at $r = 1{\cdot}9$ cm	$3{\cdot}61 > 2{\cdot}68 + 0{\cdot}76 = 3{\cdot}44$;
at $r = 1{\cdot}87$ cm	$3{\cdot}49 \approx 2{\cdot}72 + 0{\cdot}76 = 3{\cdot}48$.

We shall determine the number of coils

$$n = \frac{\lambda G r^4}{4 P R^3} = \frac{2 \times 8 \times 10^5 \times 1{\cdot}87^4}{4 \times 600 \times 11{\cdot}87^3} = 4{\cdot}86 \approx 5 \text{ coils}.$$

3.97. The coils of a helical shock absorber spring have a mean diameter of 20 cm. The spring is to be designed so that when compressed 5 cm it would absorb an amount of energy equal to 0·1 t m. Determine the diameter of the wire, and the number of coils. The allowable stress in shear is 1500 kg/cm²; the modulus G is 8×10^5 kg/cm².

Answer: 5·36 cm; 13 coils.

3.98. A helical spring is to be designed to have a stiffness of 40 kg/cm, and a total deflection (when the coils touch each other) of 4 cm. The mean coil diameter is 6 cm. The allowable shear stresses are 1400 kg/cm². Determine the diameter of the steel wire, the number of coils, and the clearance between the coils in the unloaded state.

Answer: 12·44 mm; 28 coils; 1·43 mm.

3.99. A helical spring is made of steel tube the internal diameter of which is 0·8 times the external diameter. Determine the potential energy per one kilogramme of the metal of the tube, if the mean value of the shear stresses in the cross-section of the tube (due to torsion) is 4500 kg/cm²; $G = 8 \times 10^5$ kg/cm²; $\gamma = 7{\cdot}85$ g/cm.

Answer: 16·3 kg m/kg.

3.100. A helical steel spring is to elongate 2·5 cm for each 50 kg of an axial load. The maximum extension must not exceed 12·5 cm. The diameter of the generating cylinder is to be 7·5 cm. Determine the diameter of wire, the number of coils, and the weight of the spring, if the allowable shear stress is 1400 kg/cm².

Answer: 16·6 mm; 50; 24·4 kg.

3.101. Opening of a door is resisted by a spring transmitting a couple proportional to the angle of the opened door. The door handle is 60 cm from the vertical line of the hinges. A force of 0·5 kg, gradually applied to the handle, opens the door through 10°.

What are the potential energy and torque when the door is opened through 60°?

Answer: 94·2 kgcm; 18 kgm.

3.102. A helical spring having 20 coils, with a generating cylinder 5 cm in diameter, is made of steel wire 10 mm in diameter. The spring is loaded by a force of 5 kg. Determine the potential energy of the spring.

Answer: 0·54 kgcm.

3.103. Design a helical spring absorbing 270 kgm of energy under the following conditions: (a) the maximum stress must not exceed 5600 kg/cm^2; (b) the maximum deflection of the spring is to be 34 cm; (c) the mean diameter of the coils is to be ten times the diameter of the wire of which the spring is made. Determine the diameter of the wire, the mean diameter of coils, and the number of coils.

Answer: 2·76 cm; 27·6 cm; 6 coils.

3.104. A safety valve 7·5 cm in diameter, closed by a spring, must open at a pressure of 8 atm. The valve opens only after it is lifted by 2 cm. The spring has a mean diameter 6 cm, and is made of a wire 12 mm in diameter. The clear distance between coils in the unloaded spring equals 5 mm. Determine the required number of coils, if the spring has a further capacity of deflection of 16 mm, when the valve is open. Determine also the initial compressive force, and the maximum shear stress in the material of the spring.

Answer: 12 coils; 192 kg; 3430 kg/cm^2.

3.105. Two helical springs with an identical generating cylinder are made of two wires of equal length—but one wire is of steel, and the other of bronze. The diameter of the bronze wire is 1·5 times greater than the diameter of the steel wire. Determine the ratio of the elongations and, that of the maximum shear stresses in both wires under identical loading.

Answer: 81/32; $\approx 27/8$.

3.106. Inside a helical spring of circular cross-section 2 cm in diameter another spring is placed, having the same cross-section. The

mean radius of the outer spring is 8 cm, whereas that of the inner one is 5 cm. Both springs are of the same height, and they have 10 coils each. With a load transmitted to both springs they deflect 5 cm. In which spring are the maximum shear stresses greater, and what are they? What is the load on each spring.

Answer: In the inner spring, 2810 kg/cm^2; 995 kg.

3.107. A helical steel spring of square cross-section 10 $\times$ 10 mm^2 has 12 coils with a mean radius of 5 cm. Determine the maximum force with which the spring may be loaded, if the allowable shear stress is 2800 kg/cm^2. What will the deflection of the spring be corresponding to this stress?

Answer: 112 kg; 9·4 cm.

3.108. A conical coil spring of circular cross-section 2 cm in diameter is compressed by a force P of 400 kg. The smallest radius of the coil is 4 cm, and that of the bottom coil is 10 cm. Determine the magnitude of the maximum shear stresses in the spring, and the number of coils, if the deflection of the spring is 6·5 cm.

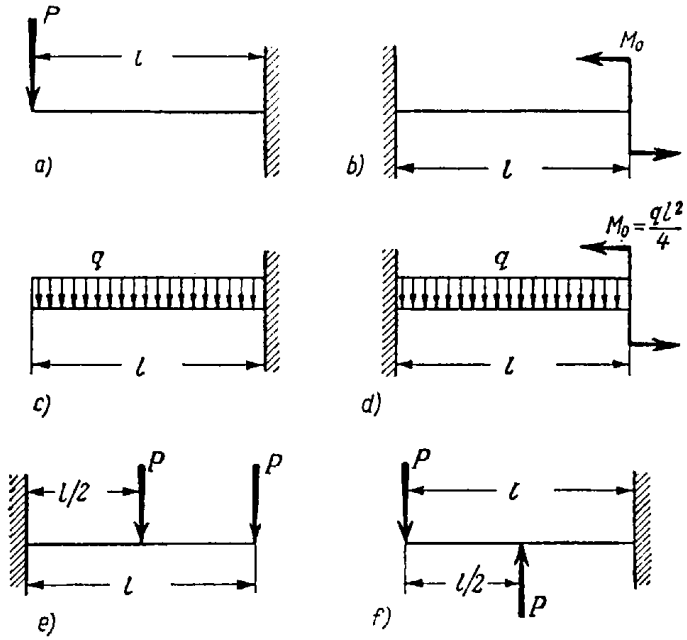
Answer: 2690 kg/cm^2; 8.

CHAPTER 4

PLANE FLEXURE

14. CONSTRUCTION OF SHEAR FORCE AND BENDING MOMENT DIAGRAMS

4.1. Construct shear force and bending moment diagrams for cantilevers loaded as shown in the figure.



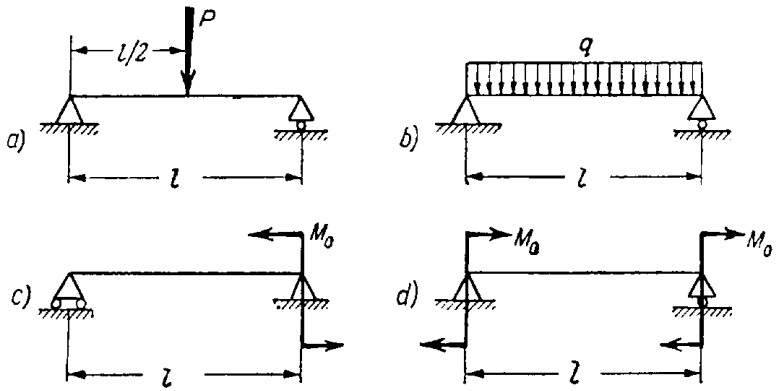
PROBLEM 4.1

Calculate the greatest numerically values of shear forces and bending moments, if $P = 2\,\mathrm{t}$, $q = 2\,\mathrm{t/m}$, $M_0 = 4\,\mathrm{tm}$ and $l = 2\,\mathrm{m}$.

Answer:

	(a)	(b)	(c)	(d)	(e)	(f)
Q (t)	2	0	4	4	4	2
M (tm)	4	4	4	2	6	2

4.2. Construct shear force and bending moment diagrams for simply supported beams loaded as shown in the figure.



PROBLEM 4.2

Find the maximum Q and M, if $P = 6$ t, $q = 2$ t/m, $M_0 = 6$ tm and $l = 3$ m.

Answer:

	(a)	(b)	(c)	(d)
Q (t)	3	3	2	4
M (tm)	4·5	2·25	6	6

4.3. A simply supported beam, with a span $l = 5a$, is loaded over a length of $4a$ by a continuous, uniformly distributed load of intensity q, and by a couple $M_0 = qa^2$, applied at distance a from the left-hand support (see figure). Construct shear force and bending moment diagrams, and calculate the maximum numerical values of these quantities.

Solution. (1) Determination of reactions. We add reactions A and B to the loading diagram (directing them upwards), and set up

the equations of equilibrium. The horizontal reaction at A is clearly equal to zero.

$$\sum M_B = 0;$$

$$(A \times 5a) - M_0 - (q \times 4a \times 2a) = 0;$$

$$A = \frac{1}{5a}(8qa^2 + M_0) = 1{\cdot}8qa;$$

$$\sum M_A = 0;$$

$$-(B \times 5a) + (q \times 4a)(2a + a) - M_0 = 0;$$

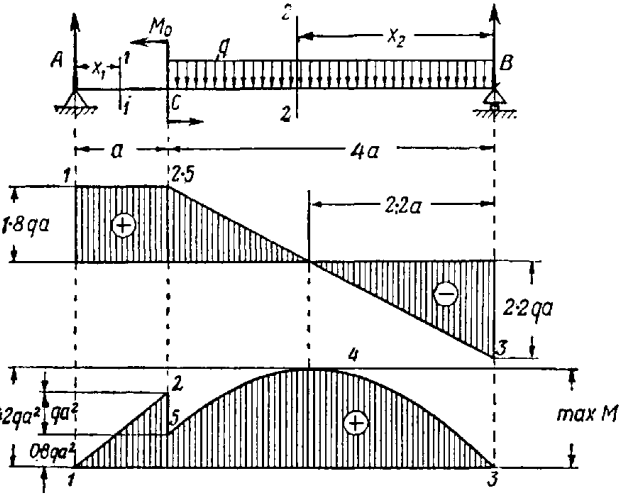
$$B = -\frac{1}{5a}(12qa^2 - M_0) = 2{\cdot}2qa.$$

To check the results we have

$$\sum Y = 0; \quad A + B - (q \times 4a) = 0;$$

$$1{\cdot}8qa + 2{\cdot}2qa - 4qa = 0.$$

The reactions have been determined correctly.



PROBLEM 4.3

(2) Derivation of expressions $Q(x)$ and $M(x)$. To obtain equations expressing the varying shear forces and bending moments it will be necessary to consider the two sections: section 1–1 between support A and point C, and section 2–2 within the limits of loaded portion CB of the beam.

It is convenient to carry out the calculation of $Q(x)$ and $M(x)$ for section 1–1 by considering the forces on the left of this section (over the length x_1). For the second section it is better to consider the right-hand portion of the beam (over the length x_2). For these reasons the origin is taken at support A for the first section, and at support B for the second section.

Expressions $Q(x)$ and $M(x)$:

for the first section	for the second section
$0 \leq x_1 \leq a$,	$0 \leq x_2 \leq 4a$,
$Q_1 = A = 1{\cdot}8qa$,	$Q_2 = -B + qx_2 = -2{\cdot}2qa + qx_2$,
$M_1 = Ax_1 = 1{\cdot}8qax_1$,	$M_2 = +Bx_2 - \frac{qx_2^2}{2} = 2{\cdot}2qax_2 - \frac{qx_2^2}{2}$.

These equations show that for the first section both diagrams consist of straight lines, whereas for the second section the bending moment diagram is represented by a parabolic curve.

(3) Construction of M and Q diagrams. To draw these diagrams for the first section of the beam it is sufficient to give the variable x_1 two values; to draw the bending moment diagram for the second section the variable x_2 should not be given less than three values. The required calculations are given in the table below:

Point No.	Value of x	Shear force $Q(x)$	Bending moment $M(x)$
1	$x_1 = 0$	$Q_1 = 1{\cdot}8qa$	$M_1 = 0$
2	$x_1 = a$	$Q_1 = 1{\cdot}8qa$	$M_1 = 1{\cdot}8qa^2$
3	$x_2 = 0$	$Q_2 = -2{\cdot}2qa$	$M_2 = 0$
4	$x_2 = 2a$	$Q_2 = -0{\cdot}2qa$	$M_2 = 2{\cdot}2qa \times 2a - \frac{q(2a)^2}{2} = 2{\cdot}4qa^2$
5	$x_2 = 4a$	$Q_2 = -2{\cdot}2qa + 4qa = 1{\cdot}8qa$	$M_2 = 2{\cdot}2qa \times 4a - \frac{q(4a)^2}{2} = 0{\cdot}8qa^2$

The Q and M diagrams are drawn, making use of the values for abscissae and ordinates given in the table. However, to obtain an accurate M diagram for the second section, it is necessary to investigate the expression of M_2 for the maximum value:

$$\frac{dM_2}{dx_2} = -Q_2 = 2{\cdot}2qa - qx_2 = 0,$$

($dM_2/dx_2 = -Q_2$, since the x_2 axis is directed from right to left).

Hence we determine the value of x_2 at which $M_2 = M_{\max}$; $x_2 = 2{\cdot}2a$.

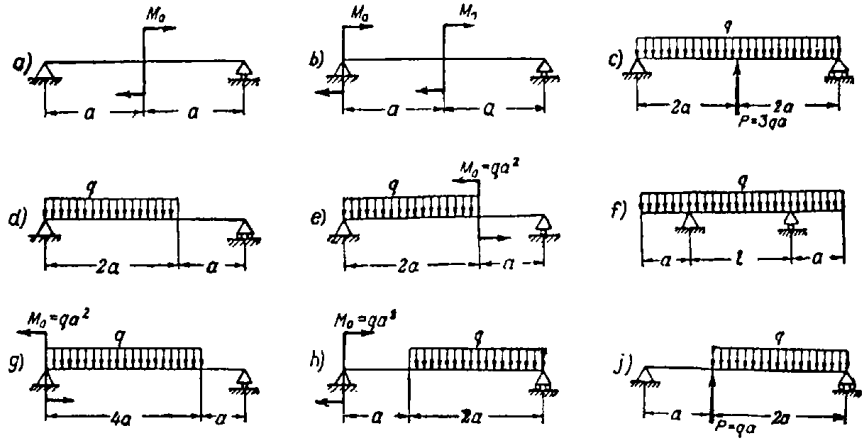
Substituting this value of x_2 in the expression for M_2, we find

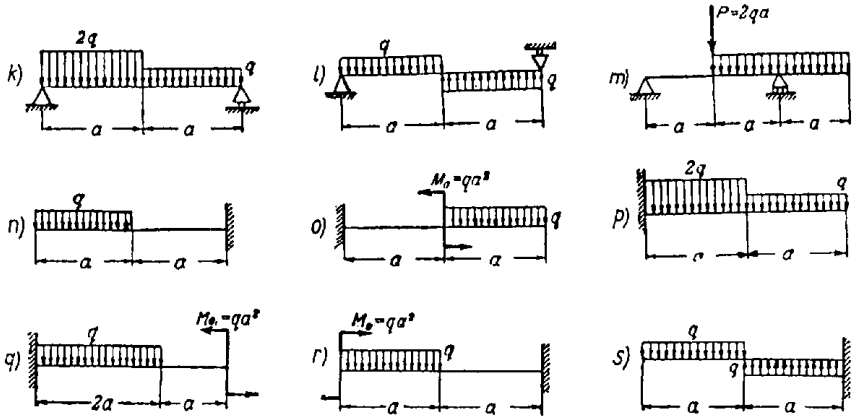
$$M_{\max} = 2{\cdot}2qa \times 2{\cdot}2a - \frac{q(2{\cdot}2a)^2}{2} = 2{\cdot}42qa^2.$$

This ordinate of the bending moment diagram corresponds to the zero value of Q.

From the diagrams it is seen that the maximum absolute values of Q and M are: $|\max Q| = 2{\cdot}2qa$ (at support B) and $\max M = 2{\cdot}42qa^2$ (at distance $2{\cdot}2a$ from the right-hand support B).

4.4. Construct Q and M diagrams for the beams shown in the figure.



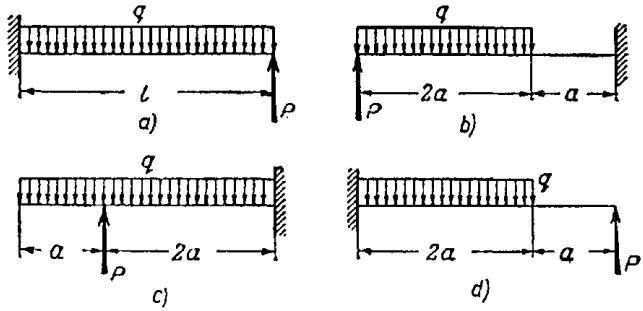


PROBLEM 4.4

4.5. At what values of force P is there a zero bending moment at the fixed ends of the cantilevers shown in the figure? Draw Q and M diagrams at these values of force P.

Answer:

	(a)	(b)	(c)	(d)
$P =$	$q\frac{l}{2}$	$\frac{4}{3}qa$	$\frac{4}{9}qa$	$\frac{2}{3}qa$



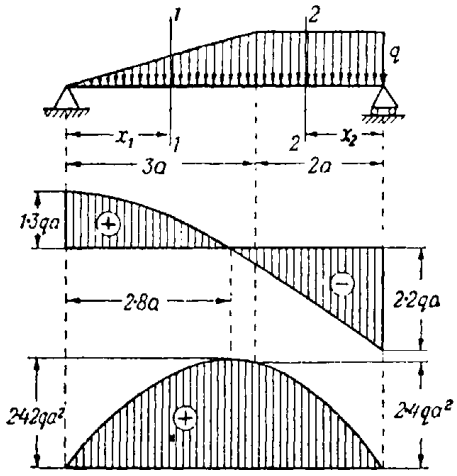
PROBLEM 4.5

4.6. Investigate the variation in shear force and bending moment of the beam shown in the figure, and draw M and Q diagrams.

Solution. From the equations of equilibrium we determine the values of reactions A and B.

Both the reactions are directed upwards and are respectively:

$$A = 1{\cdot}3qa; \quad B = 2{\cdot}2\,qa.$$



PROBLEM 4.6

We shall set up the equations for Q and M in each section of the beam.

Section 1–1. $Q_1 = A - \omega_1$, where ω_1 is the load diagram area on the left from section 1–1, and equal to the area of a triangle having a base x_1 and a height $q_1 = q(x_1/3a)$. Consequently,

$$Q_1 = 1{\cdot}3qa - \frac{qx_1^2}{6a};$$

$$M_1 = Ax_1 - \omega_1 \frac{x_1}{3} = 1{\cdot}3qax_1 - \frac{qx_1^3}{18}.$$

Section 2–2. Considering the part on the right from the section we have

$$Q_2 = -B + qx_2 = -2{\cdot}2qa + qx_2;$$

$$M_2 = Bx_2 - \frac{qx_2^2}{2} = 2{\cdot}2qax_2 - \frac{qx_2^2}{2}.$$

Thus the shear force in the first section varies as a parabola, whereas in the second section it varies linearly. The law of the bending moment in the first section is that of a cubic parabola; in the second section the law is that of a quadratic parabola.

To draw the Q and M diagrams we compile the following table of values of $Q(x)$ and $M(x)$:

x	$Q(x)$	$M(x)$
$x_1 = 0$	$Q_1 = 1{\cdot}3qa$	$M_1 = 0$
$x_1 = 2a$	$Q_1 = 1{\cdot}96qa$	$M_1 = 2{\cdot}16qa^2$
$x_1 = 3a$	$Q_1 = -0{\cdot}2qa$	$M_1 = 2{\cdot}4qa^2$
$x_2 = 0$	$Q_2 = -2{\cdot}2qa$	$M_2 = 0$
$x_2 = a$	—	$M_2 = 1{\cdot}7qa^2$
$x_2 = 2a$	$Q_2 = -0{\cdot}2qa$	$M_2 = 2{\cdot}4qa^2$

The Q and M diagrams, drawn using the data in the table, are shown in the figure.

To determine the maximum absolute values of the quantities M_1 and M_2 it is necessary to investigate the corresponding equations for the maximum. The condition for the maximum value of a bending moment is expressed by the formula

$$\frac{\mathrm{d}M}{\mathrm{d}x} = Q(x) = 0.$$

For the first section

$$Q_1 = 1{\cdot}3qa - \frac{qx_1^2}{6a} = 0;$$

hence

$$x_1 = a\sqrt{7{\cdot}8} \approx 2{\cdot}8a,$$

and the maximum value of M will be

$$\max M_1 = 1{\cdot}3qa \times 2{\cdot}8a - \frac{q}{18a}(2{\cdot}8a)^3 = 2{\cdot}42qa^2.$$

For the second section

$$Q_2 = -2{\cdot}2qa + qx_2 = 0,$$

whence

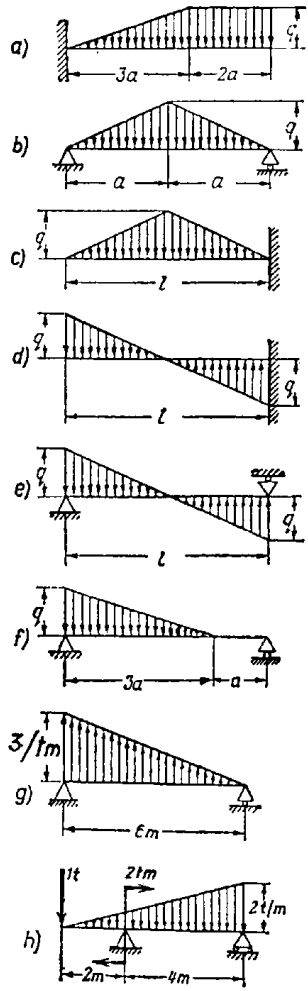
$$x_2 = 2{\cdot}2a$$

(the length of the section being $2a$). Thus the parabola, which expresses the function M_2, has its maximum beyond the limits of the section considered.

The bending moment has the value $\max M = 2{\cdot}42qa^2$ for a section located at $2{\cdot}8a$ from the left-hand support.

Referring to the shear force, it will be seen from the diagram that the maximum absolute value occurs at the right-hand support B. It is equal to the reaction of this support, namely $|\max Q| = 2{\cdot}2qa$.

4.7. Construct shear force and bending moment diagrams for the beams represented schematically in the figure, and calculate the maximum values of Q and M.



PROBLEM 4.7

Answer:

(a) $Q = 3{\cdot}5qa$; $M = -11qa^2$;

(b) $Q = 0{\cdot}5qa$; $M = \frac{1}{3}\,qa^2$;

(c) $Q = 0{\cdot}5ql$; $M = -0{\cdot}25ql^2$;

(d) $Q = -0{\cdot}25ql$; $M = -\frac{1}{6}\,ql^2$;

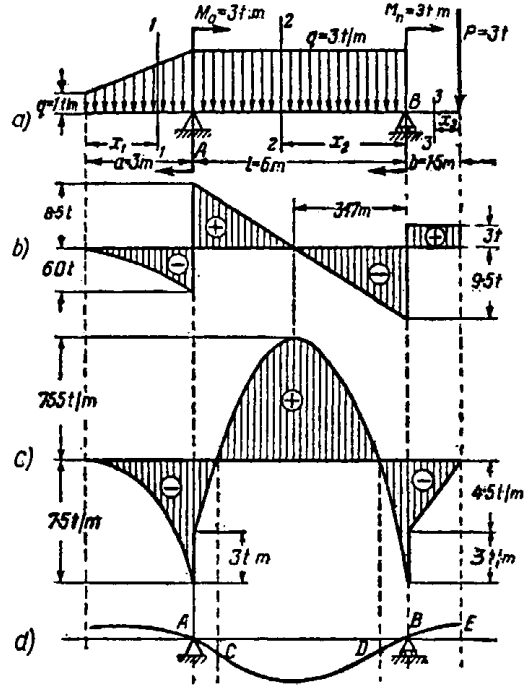
(e) $Q = \frac{1}{16}\,ql$; $M = \frac{1}{216}\,ql^2$;

(f) $Q = \frac{9}{8}\,qa$; $M = \frac{3}{8}\,qa^2$;

(g) $Q = 6$ t; $M = 6{\cdot}93$ tm;

(h) $Q = 3{\cdot}0$ t; $M = 2{\cdot}53$ tm.

4.8. Construct Q and M diagrams for a simply supported beam with overhangs, loaded as shown in the figure (a). Sketch also the



PROBLEM 4.8

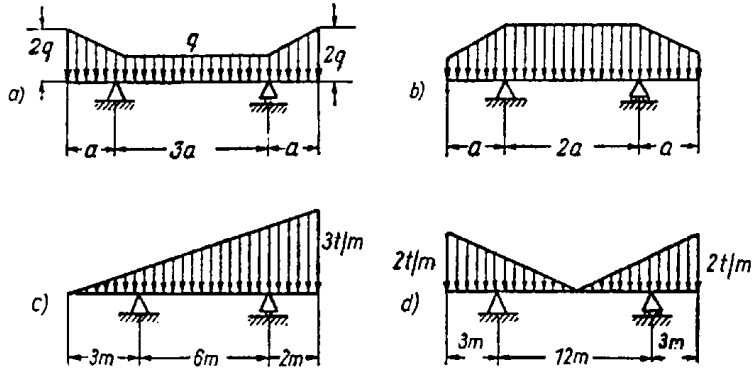
deflection curve for the beam, being guided by the bending moment diagram.

Answer: The Q and M diagrams are shown in the figures (b) and (c).

An approximate shape of the deflection curve is shown in the figure (d). The left-hand portion of the beam, up to point C, is bent convexly upwards, which corresponds to the negative value of M. The points C and D, corresponding to the zero points of the M diagram, are the points of inflexion. Over the part CD the beam is bent convexly downwards.

The right-hand end DE of the beam has an upward convexity.

4.9. Draw shear force and bending moment diagrams for the beams with overhangs shown in the figure. Indicate also the character of the deflection curve for the beam, and determine the point of inflexion (in cases where they exist).



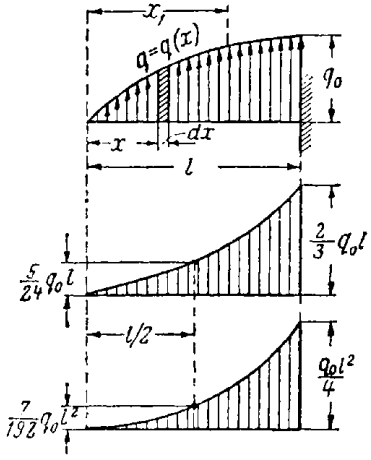
PROBLEM 4.9

4.10. Construct Q and M diagrams for a cantilever subjected to a continuous load directed upwards. The intensity of the load varies according to the equation of a parabola

$$q(x) = q_0 \left(\frac{2x}{l} - \frac{x^2}{l^2} \right).$$

Solution. Bearing in mind that an elemental load on an infinitely small section of the beam, having a length $\mathrm{d}x$, is given by

$$\mathrm{d}\omega = q(x)\mathrm{d}x = q_0\left(\frac{2x}{l} - \frac{x^2}{l^2}\right)\mathrm{d}x,$$



PROBLEM 4.10

we shall find Q and M for a section located at a distance x_1 from the left-hand end of the beam:

$$Q = \int_0^{x_1} q(x)\mathrm{d}x = \frac{q_0}{l}\int_0^{x_1}\left(2x - \frac{x^2}{l}\right)\mathrm{d}x = \frac{q_0 x_1^2}{l} - \frac{q_0 x_1^3}{3l^2};$$

$$M = \int_0^{x_1} q(x)\mathrm{d}x(x_1 - x) = \frac{q_0}{l}\int_0^{x_1}\left(2x - \frac{x^2}{l}\right)(x_1 - x)\mathrm{d}x =$$

$$= \frac{q_0}{l}\left[2x_1\int_0^{x_1} x\mathrm{d}x - 2\int_0^{x_1} x^2\mathrm{d}x - \frac{x_1}{l}\int_0^{x_1} x^2\mathrm{d}x + \frac{1}{l}\int_0^{x_1} x^2\mathrm{d}x\right] =$$

$$= \frac{q_0 x_1^3}{3l} - \frac{q_0 x_1^4}{12l^2}.$$

To draw the diagrams we shall compile a table of values for Q and M:

x_1	$Q(x_1)$	$M(x_1)$
0	$Q_1 = 0$	$M_1 = 0$
$\frac{l}{2}$	$Q_1 = \frac{q_0 l}{4} - \frac{q_0 l}{24} = \frac{5}{24} q_0 l$	$M_1 = \frac{q_0 l^2}{24} - \frac{q_0 l^2}{8 \times 24} = \frac{7}{192} q_0 l^2$
l	$Q_1 = q_0 l - \frac{q_0 l}{3} = \frac{2}{3} q_0 l$	$M_1 = \frac{q_0 l^2}{3} - \frac{q_0 l^2}{12} = \frac{q_0 l^2}{4}$

The diagrams of Q and M are shown in the figure. We note that the maximum ordinate of the Q diagram expresses the value of the load diagram area, i.e. the area of the parabolic triangle: $\omega = \frac{2}{3} q_0 l$. The maximum ordinate of the M diagram is numerically equal to the moment of this area about the point of fixing: $S = \frac{q_0 l^2}{4}$. Hence the distance from the clamped end to the centre of gravity of the parabolic load is

$$x_c = \frac{S}{\omega} = \frac{q_0 l^2}{4 \times \frac{2}{3} q_0 l} = \frac{3}{8} l,$$

or from the free end

$$x_c' = \frac{5}{8} l.$$

4.11. Construct Q and M diagrams, and find the maximum values of Q and M for a simply supported beam of span l, subjected to a continuous load varying as a parabola defined by the equation $q(x) = (4/l^2) q_0 (lx - x^2)$.

Answer: $\max Q = \frac{q_0 l}{3}$; $\max M = \frac{5}{48} q_0 l^2$.

4.12. Construct diagrams of shear forces and bending moments, and calculate the maximum values of the quantities Q and M for

beams loaded by a continuous load having an intensity $q = q(x)$ as follows:

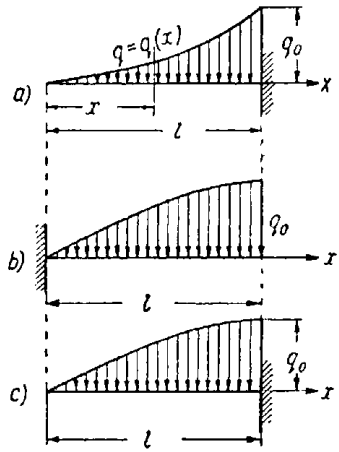
(a) $q(x) = q_0 \frac{x^2}{l^2}$; (b) $q(x) = q_0 \left(\frac{2x}{l} - \frac{x^2}{l^2}\right)$;

(c) $q(x) = q_0 \sin \frac{\pi x}{2l}$.

Answer:

(a) $Q = -\frac{q_0 l}{3}$; (b) $Q = \frac{2}{3} q_0 l$; (c) $Q = -\frac{2}{\pi} q_0 l$;

$M = -\frac{q_0 l^2}{12}$; $M = -\frac{5}{12} q_0 l^2$; $M = -\frac{2\pi - 4}{\pi^2} q_0 l^2$.



PROBLEM 4.12

4.13. Construct Q and M diagrams for the beam on four supports, having intermediate hinges. The dimensions are as follows:

$a = 2$ m; $l_1 = 4$ m; $P = 2$ t;

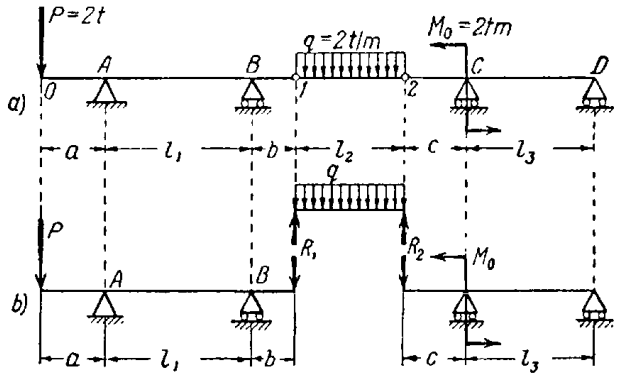
$b = 1$ m; $l_2 = 3$ m; $q = 2$ t/m;

$c = 3$ m; $l_3 = 4$ m; $M = 2$ tm.

Instruction. It will be convenient to visualize the beam as consisting of three independent beams: the middle one suspended

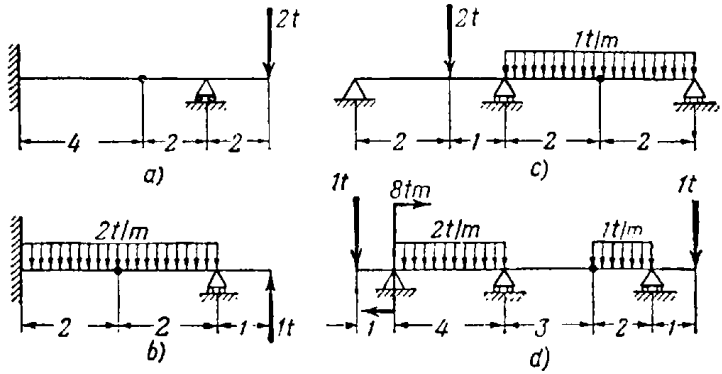
from the overhangs b and c by means of the hinges 1 and 2, and the other two beams supported simply and having overhangs (O–1 and 2–D). The loads transmitted through the hinges from the suspended beam to the overhangs of the simply supported beams are equal to the reactions R_1 and R_2 of the suspended beam but are opposite in direction (see diagram (b).) The Q and M diagrams may now be constructed for each of the beams taken separately.

Answer: $\min M = -11$ tm; $\max Q = 3$ t.



PROBLEM 4.13

4.14. Construct Q and M diagrams for the beams with hinges represented diagrammatically in the figure, and calculate the maximum Q and M. The dimensions given in the figure are in metres.



PROBLEM 4.14

Answer: The maximum values of Q and M respectively are:

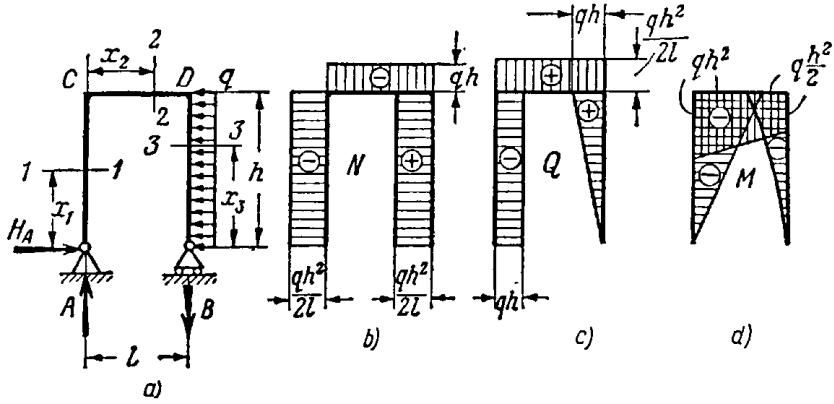
(a) $M = 8$ tm; $Q = 2{\cdot}0$ t;

(b) $M = -9{\cdot}0$ tm; $Q = -6{\cdot}5$ t;

(c) $M = -4{\cdot}0$ tm; $Q = 3{\cdot}0$ t;

(d) $M = 7{\cdot}0$ tm; $Q = -6{\cdot}62$ t.

4.15. Construct axial force, shear force and bending moment diagrams for the bent bar shown in the figure (a).



PROBLEM 4.15

Solution. We determine the reactions from the equations:

$$\sum X = 0; \quad H_A - qh = 0, \quad \text{whence} \quad H_A = qh;$$

$$\sum Y = 0; \quad A - B = 0, \quad \text{or} \quad A = B;$$

$$\sum M_B = 0; \quad Al - \frac{qh^2}{2} = 0, \quad \text{whence} \quad A = \frac{qh^2}{2l} = B.$$

(a) Construction of axial force diagram. The sum of the components along the axis of the rod, of external forces applied to a part sectioned off, is called the axial (or normal) force.

For section 1–1, considering the lower part of the upright AC, we have

$$N_1 = -A = -\frac{qh^2}{2l}.$$

The minus sign is taken because the force A causes compression in the upright; the value of N_1 remains constant along the whole length of the upright.

For section 2–2, considering the left-hand portion of the rod, we have

$$N_2 = -H_A = -qh \quad \text{(compression)}.$$

Finally, for section 3–3 of the right-hand upright (considering the lower part) we have

$$N_3 = B = \frac{qh^2}{2l} \text{ (tension)}.$$

In the diagram (figure (b)) the positive values of N are plotted inside, whereas the negative values are plotted outside the frame.

(b) Construction of shear force diagram. The shear forces in sections 1–1, 2–2, 3–3 are respectively

$$Q_1 = -H_A = -qh;$$

$$Q_2 = A = \frac{qh^2}{2l};$$

$$Q_3 = qx_3 \quad \text{and} \quad \max Q_3 = qh.$$

The shear force, Q, will be counted positive if the lower part of an upright is pushed to the left, or if the left-hand part of a horizontal is pushed upward. In the diagram (figure (c)) the values of Q plotted are in the direction of shear in the lower parts of the uprights, and in the left-hand part of the horizontal member.

(c) Construction of bending moment diagram. The equations of the bending moments are:

$$M_1 = -H_A x_1 = -qhx_1; \quad M_2 = Ax_2 - H_A h = \frac{qh^2}{2l} x_2 - qh^2;$$

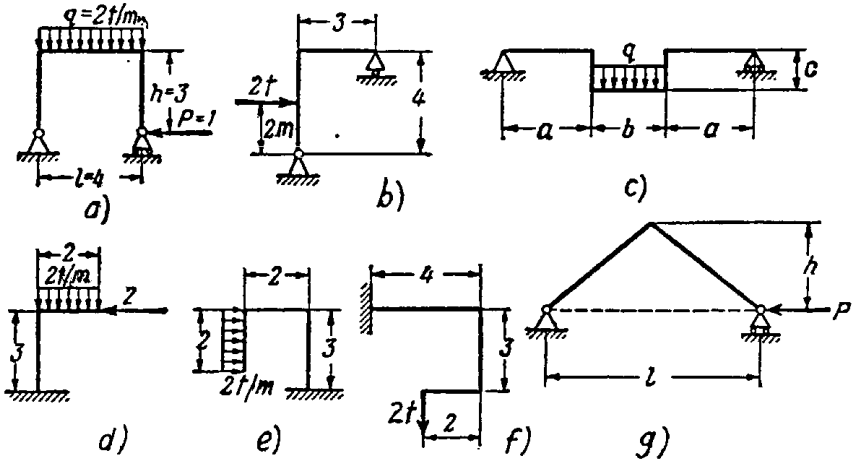
$$M_3 = -\frac{qx_3^2}{2}.$$

The maximum absolute values are obtained for the corners C or D:

$$M_C = -qh^2; \quad M_D = -\frac{qh^2}{2}.$$

The bending moment diagram is constructed, plotting all the ordinates on the side of the compression fibre, as it was done for beams.

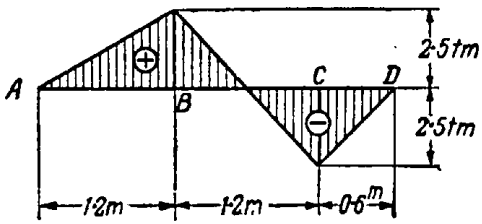
4.16. Construct N, Q, and M diagrams for the bent bars represented in the figure. The dimensions are given in metres.



PROBLEM 4.16

4.17. A bending moment diagram for a beam mounted on hinged supports at points A and C is shown in the figure. It is required to draw the load and shear force diagrams, and to find the reactions.

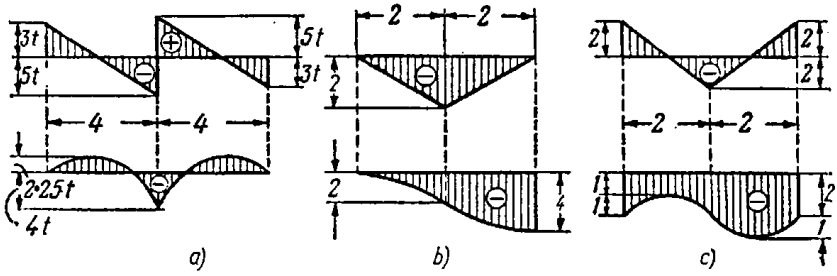
Answer: $\max Q = 4{\cdot}17$ t.



PROBLEM 4.17

4.18. Draw the loading diagrams for the beams, and determine the magnitude of the applied loads from the shear force and bending

moment diagrams shown in the figure. The lengths are indicated in metres, Q in t and M in tm.



PROBLEM 4.18

4.19. The beam ABCDE is mounted on hinged supports at points A and D. AB = 0·6 m; BC = 0·6 m; CD = 1·8 m; DE = 1 m. The beam is loaded by a concentrated force P of 2·5 t at point B. Determine what concentrated force P must be applied at point E for the bending moment at point C to be zero. Construct Q and M diagrams: (1) with, and (2) without the load at point E taken into account.

Answer: $P = 2{\cdot}25$ t.

4.20. A beam with a length of 10 m carries a load of 20 t uniformly distributed over its length. It rests on three identical hydraulic jacks which are under the same internal pressure and placed at distances of 2 m, 5 m and 8 m from one end of the beam.

(a) Construct shear force and bending moment diagrams.

(b) Leaving the middle jack fixed and displacing the end ones symmetrically, indicate in the diagrams the results of their displacement to distances of 1 m and 9 m from one end.

Answer: (a) $M_{\min} = -5$ tm; $Q_{\max} = 4$ t;
(b) $M_{\max} = 4{\cdot}45$ tm; $Q_{\max} = 4{\cdot}67$ t.

4.21. A beam loaded over its whole length by a uniformly distributed load has a span l and two overhangs with a length a each. Determine the ratio l/a at which the bending moment at the middle of the span will be zero, and the ratio l/a at which the value of the

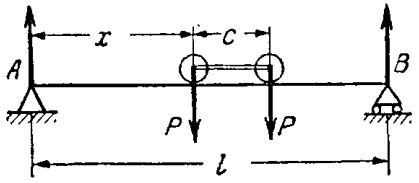
maximum bending moment in the span will be equal to the bending moment at the support.

Answer: $a = \frac{l}{2}$; $a = \frac{\sqrt{2}}{4} l$.

4.22. A trolley, transmitting a thrust of P t through each of its two axles, travels along the beam of an overhead crane. How must the trolley be positioned in order to obtain the maximum value for the bending moment?

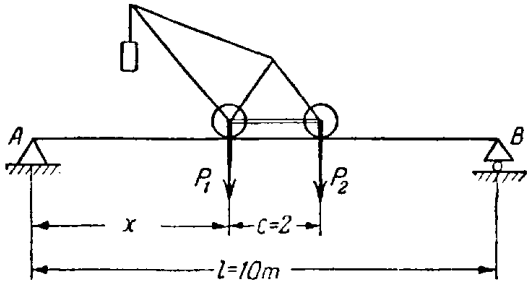
Instruction. Denoting the distance from the left-hand support to the load P by x we determine the magnitudes of the reactions as functions of x. The bending moment at the point of application of the force will be found to be a function of x of the second degree. Investigating M for a maximum we shall find the required value of x.

Answer: $x = \frac{l}{2} - \frac{c}{4}$.



PROBLEM 4.22

4.23. The biaxial carriage of a luffing crane travels along two beams under the crane. The distance c between the axles is 2 m. The span l of the beams is 10 m. The thrusts from the crane wheels

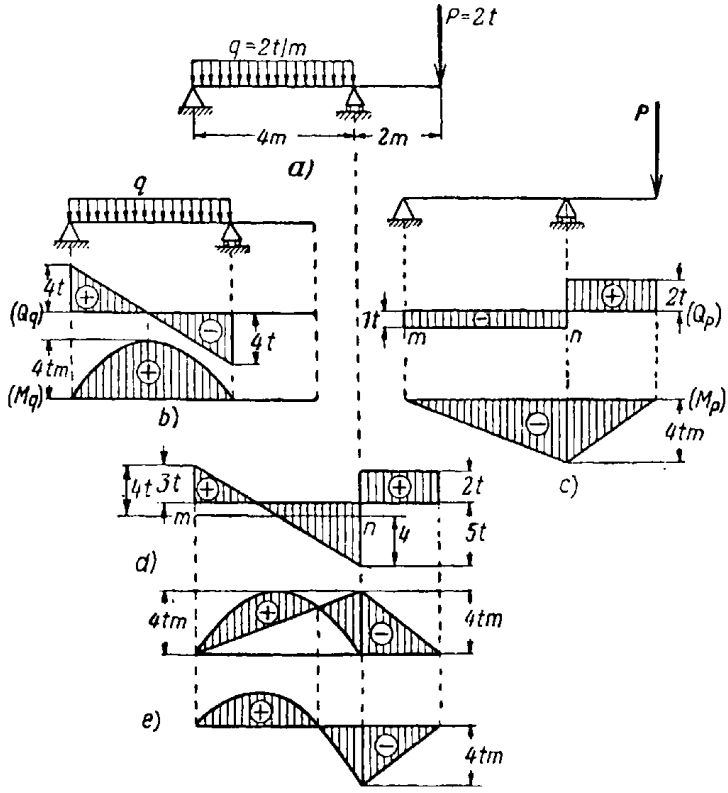


PROBLEM 4.23

to each beam are: $P_1 = 5$ t and $P_2 = 1$ t. Find the critical position for the loads and calculate the max M.

Answer: $x = \frac{l}{2} - \frac{c}{12}$.

4.24. Using the method of superposition construct Q and M diagrams for the beam shown in figure (a).



PROBLEM 4.24

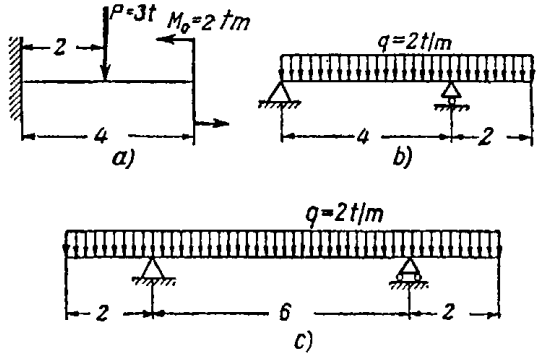
Solution. We consider the beam as consisting of two parts; one loaded only by the continuous load, and the other loaded only by the force at its end. The Q and M diagrams for each of them are shown in figures (b) and (c). To obtain the resultant diagrams the graphs having the same sign are subtracted, whereas those of different signs are superimposed on each other. These results are

shown in figure (d). The bottom line mn of the Q_P diagram is made to coincide with the zero line of the Q_q diagram; thereby the addition of both diagrams is carried out automatically.

The resultant M diagram shown in the figure (d) is reproduced with a horizontal zero line in figure (e).

4.25. Using the method of superposition plot Q and M diagrams for the beams considered in Problem 4.5.

4.26. Applying the method of superposition plot bending moment diagrams for the beams shown in the figure. The dimensions given are in metres.



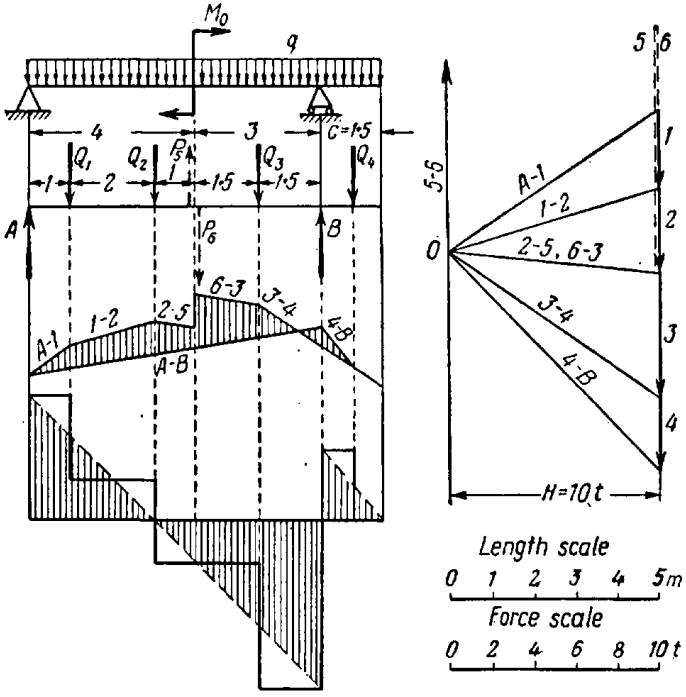
PROBLEM 4.26

4.27. A beam on two supports with a span $l = 7$ m and an overhang $c = 1{\cdot}5$ m is loaded by a continuous uniformly distributed load $q = 2$ t/m and a couple $M_0 = 6$ tm, applied at a distance $a = 4$ m from the left-hand support A. Employing a graphical method plot Q and M diagrams.

Solution. Having chosen a length scale (see figure) we draw the diagram of the beam. We replace the continuous load by a number of concentrated forces, having divided it for example into four parts: $Q_1 = Q_2 = 4$ t, $Q_3 = 6$ t and $Q_4 = 3$ t. We replace the couple by two vertical forces P_5 and P_6 having an infinitely large magnitude and an infinitely small lever arm between them.

Having chosen a force scale and a polar distance H we plot the force and funicular polygons. In the force polygon the line 5–6 is parallel to the forces, since the line joining the pole O to the end

of the force P intersects the direction of the force at infinity. In the construction of the funicular polygon its side parallel to the line 5–6 is vertical and equal to the quantity $M_0 = 6\,\text{t m}$. Since $M = H \times \eta = 6\,\text{t m}$, and for the construction H is taken as 10 t, the distance η set off upwards ($M_0 > 0$) is 0·6 m.



PROBLEM 4·27

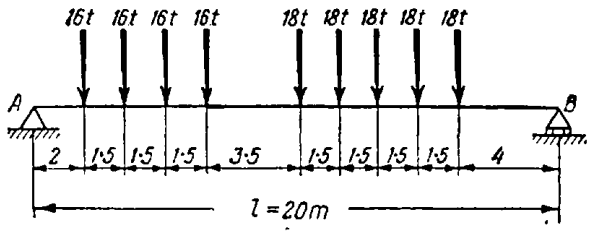
Increasing the number of the concentrated forces Q, the funicular polygon in the limit is transformed into a funicular curve, and the stepped Q diagram is transformed into the inclined line shown dotted in the figure.

4.28. Using the graphical method construct bending moment diagrams for the beams considered in Problems 4.3 and 4.8.

4.29. Employing the graphical method plot the Q and M diagrams for the main beam of a bridge loaded by a system of concentrated

forces, representing thrusts due to locomotive wheels. The dimensions shown are in metres.

Answer: $\max Q = 81{\cdot}9$ t; $\max M = 451$ tm.



PROBLEM 4.29

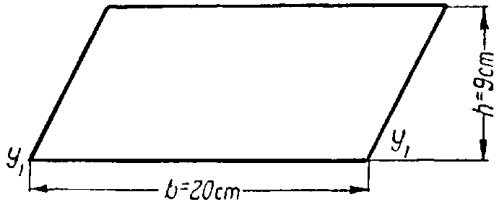
15. SECOND MOMENTS OF AREA OF PLANE FIGURES

4.30. Calculate the second moment of area of a rectangle having sides $b = 12$ cm and $h = 20$ cm, about the y_1 axis passing through the base. Calculate the product second moment of area about the axes coinciding with the sides of the rectangle.

Answer: $J'_y = 32{,}000$ cm^4; $J'_{yz} = \dfrac{b^2h^2}{4} = 14{,}400$ cm^4.

4.31. Find the second moment of area of a parallelogram (see figure) about the y_1 axis.

Answer: $J'_y = 4860$ cm^4.

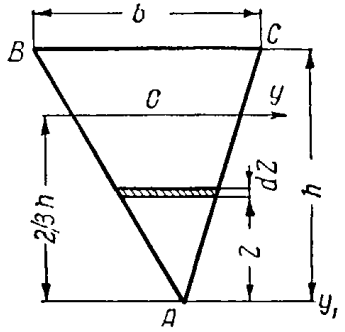


PROBLEM 4.31

4.32. Calculate the second moment of area of triangle ABC (see figure) with respect to the y_1 axis passing through the apex A, parallel to the base BC. Find also the second moment of area of the triangle with respect to the centroidal y_0 axis.

Instruction. Taking an elemental strip with an area $\mathrm{d}F = b(z)\mathrm{d}z$, and expressing the width $b(z)$ of the strip from the similarity of the triangles, calculate $J'_y = \int_F z_1^2 \,\mathrm{d}F$. For the determination of the second moment of area about the centroidal y axis use the theorem of parallel axes.

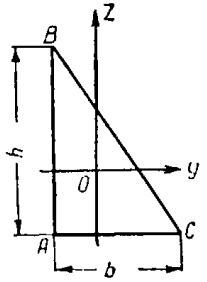
Answer: $J'_y = \dfrac{bh^3}{4}$; $J_y = \dfrac{bh^3}{36}$.



PROBLEM 4.32

4.33. Find the second moments of area and the product second moment of area of triangle ABC with respect to the centroidal axes Oy and Oz parallel to the sides forming the right angle (see figure). Calculate also the second moment of area of the triangle about the base AC.

Answer: $J^0_{yz} = -\dfrac{b^2h^2}{72}$; $J_{AC} = \dfrac{bh^3}{12}$.

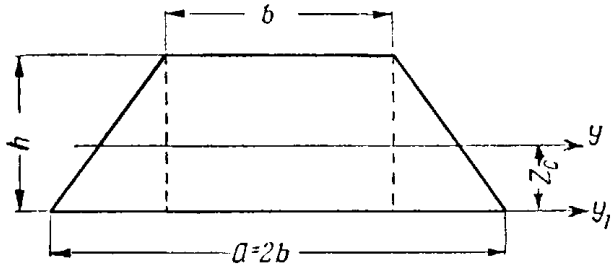


PROBLEM 4.33

4.34. Calculate the second moment of area of an equal-sided trapezium (see figure) about the centroidal y axis parallel to the base.

Instruction. It is recommended to divide the figure into parts for which known formulae exist, for example, into a rectangle and two triangles.

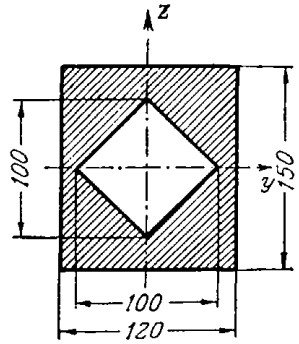
Answer: $J_y = \dfrac{13}{108} bh^3 \approx 0{\cdot}12bh^3$.



PROBLEM 4.34

4.35. Determine the second moment of area about the neutral axis y for the cross-section shown in the figure. The dimensions shown are in mm.

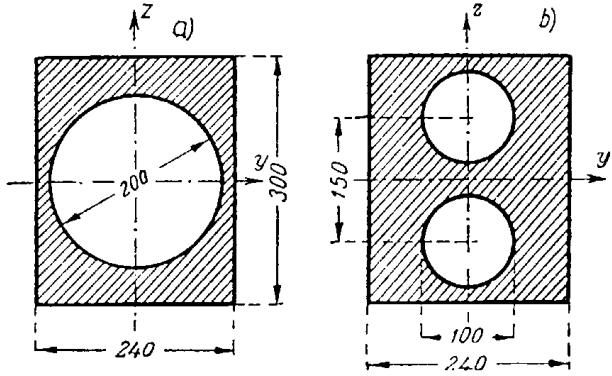
Answer: $J_y = 3167$ cm^4.



PROBLEM 4.35

4.36. The cross-section of a hollow beam is shown in the figure (a). How will the area and the second moment of area (J_y) change, if the single hole 20 cm in diameter is replaced by two holes, each 10 cm in diameter, located as shown in figure (b)?

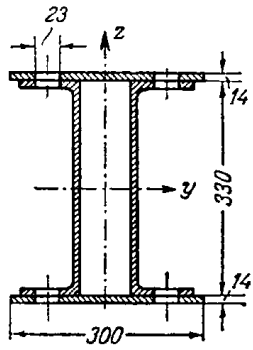
Answer: The area increases by 38·6%; the second moment of area decreases by 2%.



PROBLEM 4.36

4.37. The cross-section of a beam consists of two No. 33 b channels strengthened by two 14 × 300 mm² plates fixed to the flanges. Determine the second moment of area of the cross-section about the centroidal y axis, taking into account the rivet holes with $d = 23$ mm (see figure). Determine the reduction in the second moment of area of the cross-section (as a percentage of the second moment of area of the section without rivet holes).

Answer: $J = 35{,}200$ cm⁴; $\Delta J = 16{\cdot}5\%$.

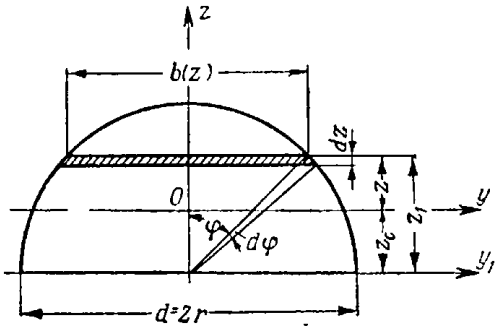


PROBLEM 4.37

4.38. Find the second moments of area of a semicircle about the principal centroidal axes (see figure).

Answer: $z_C = \dfrac{4r}{3\pi}$; $J_y = \dfrac{\pi r^4}{8}\left(1 - \dfrac{64}{9\pi^2}\right) \approx 0{\cdot}11\,r^4$;

$$J_z = J'_y = \frac{1}{2}\,J = \frac{\pi r^4}{8}.$$



PROBLEM 4.38

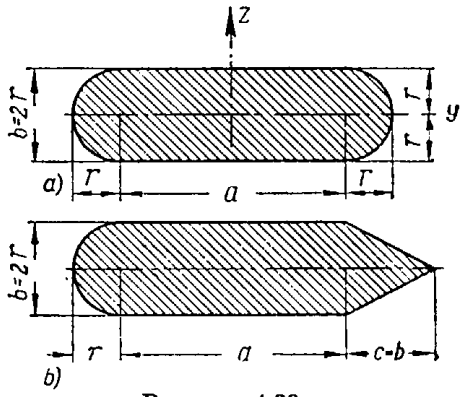
4.39. Find the principal second moments of area of the cross-sections of bridge piers shown in the diagram.

Answer: For (a) $J_y = \dfrac{ab^3}{12} + \dfrac{\pi b^4}{64}$;

$$J_z = \frac{ab}{12}(a^2 + 2ab^2) + \frac{\pi b^2}{64}(b^2 + 4a^2);$$

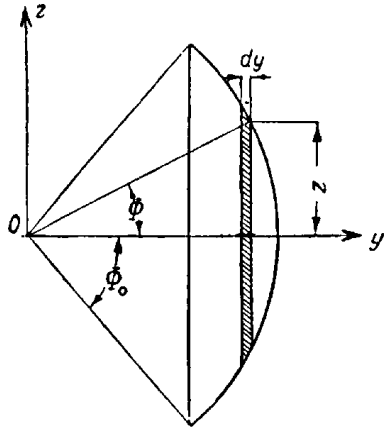
for (b) $J_y = \dfrac{ab^3}{12} + \dfrac{\pi r^4}{8} + \dfrac{cb^3}{48} =$

$$= \frac{r^3}{24}(16a + 3\pi r + 8r).$$



PROBLEM 4.39

4.40. Calculate the second moments of area of a circular segment with a radius a, with respect to the y axis (see figure).



PROBLEM 4.40

Solution. The second moment of area of an elemental strip having a height $2z$ and a base $\mathrm{d}y$ is

$$\mathrm{d}J_y = [\mathrm{d}y(2z)^3]/12 = (2/3)\, z^3 \mathrm{d}y.$$

Transformed into polar co-ordinates, we have

$$y = r\cos\Phi; \quad z = r\sin\Phi; \quad \mathrm{d}y = -r\sin\Phi;$$

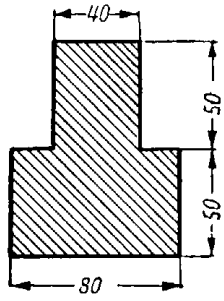
$$\mathrm{d}J_y = -\frac{2}{3}\, r^4 \sin\Phi\, \mathrm{d}\Phi.$$

Noting that during the variation of y from $y = r\cos\Phi_0$ to $y = r$ the angle Φ varies within the limits from Φ_0 to zero, we have

$$J_y = -\frac{2}{3}\, r^4 \int_{\Phi_0}^{0} \sin^4\Phi\, \mathrm{d}\Phi = \frac{r^4}{48}\,(12\Phi_0 - 8\sin 2\Phi_0 + \sin 4\Phi_0).$$

4.41. Find the second moment of area of the T-section represented in the figure, about the centroidal axis Oy parallel to the base (dimensions given are in millimetres).

Answer: $J_y \approx 458$ cm^4.

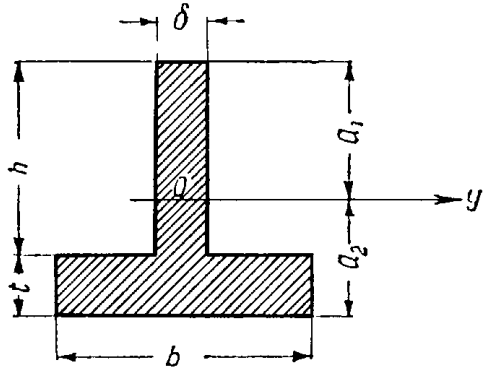


PROBLEM 4.41

4.42. Determine the law according to which the second moment of area of a T-section about the centroidal axis (see figure) will vary, if the flange thickness t is increased without altering its width b, and maintaining the web dimensions δ and h?

Answer: $$S_y = \frac{bt^2}{2} + \frac{b\delta}{2}(h + 2t);$$

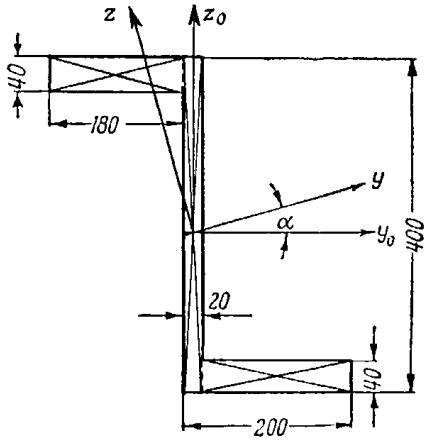
$$J_y^0 = \frac{1}{3}[\delta a_1^3 + b a_2^3 - (b - t)(a_2 - t)^3].$$



PROBLEM 4.42

4.43. Find the position of the principal centroidal axes, and calculate the value of the principal second moments of area and the maximum section modulus of the Z-section shown in the figure (dimensions given are in millimetres).

Solution. The centroid of the cross-section coincides with the mid-point of the web. Through the centroid we draw two mutually perpendicular centroidal axes so that they would be parallel to the sides of the rectangles into which it is convenient to divide the cross-section.



PROBLEM 4.43

We find the second moments of area about the centroidal axes y_0 and z_0:

$$J_y^0 = \frac{2 \times 40^3}{12} + 2\left[\frac{18 \times 4^3}{12} + (18 \times 4 \times 18^2)\right] = 57{,}500 \text{ cm}^4;$$

$$J_z^0 = \frac{2^3 \times 40}{12} + 2\left[\frac{4 \times 18^3}{12} + (4 \times 18 \times 10^2)\right] = 18{,}320 \text{ cm}^4;$$

$$J_{yz}^0 = -(2 \times 18 \times 4 \times 18 \times 10) = -25{,}900 \text{ cm}^4.$$

The angle of inclination of the principal axes to the axes y_0 and z_0 will be

$$\tan 2\alpha = \frac{2J_{yz}^0}{J_z^0 - J_y^0} = \frac{-(2 \times 25{,}900)}{18{,}320 - 57{,}500} \approx 1{\cdot}322;$$

$$2\alpha = 52^0 50' \quad \text{and } \alpha = 26^\circ 25'; \quad \sin \alpha = 0{\cdot}445;$$

$$\cos \alpha = 0{\cdot}896; \quad \sin 2\alpha = 0{\cdot}797; \quad \cos 2\alpha = 0{\cdot}604.$$

The principal second moments of area are determined by the formulae:

$$J_y = J_y^0 \cos^2 \alpha + J_z^0 \sin^2 \alpha - J_{yz}^0 \sin 2\alpha;$$

$$J_z = J_y^0 \sin^2 \alpha + J_z^0 \cos^2 \alpha + J_{yz}^0 \sin 2\alpha.$$

Inserting the numerical values in these equations we shall obtain

$$J_y = (57{,}500 \times 0{\cdot}896^2) + (18{,}320 \times 0{\cdot}445^2) - [(-25{,}900) \times 0{\cdot}797] = 70400 \text{cm}^4;$$

$$J_z = (57{,}500 \times 0{\cdot}445^2) + (18{,}320 \times 0{\cdot}896^2) - (25{,}900 \times 0{\cdot}797) \approx 5420 \text{ cm}^4.$$

Check.

$$J_y + J_z = 70{,}400 + 5420 = 75{,}820 \text{ cm}^4;$$

$$J_y^0 + J_z^0 = 57{,}500 + 18{,}320 = 75{,}820 \text{ cm}^4;$$

$$J_{yz} = \frac{J_y^0 - J_z^0}{2} \sin 2\alpha + J_{yz}^0 \cos 2\alpha = \left(\frac{57{,}500 - 18{,}320}{2} \times 0{\cdot}797\right) - (25{,}900 \times 0{\cdot}604) \approx 0.$$

The maximum distance from the y axis to the boundary of the flange is

$$z_{\max} = z_0 \cos \alpha - y_0 \sin \alpha,$$

where the distance from the centroidal axes y_0 and z_0 to the furthermost point are respectively:

$$y_0 = -19 \text{ cm}; \qquad z_0 = 20 \text{ cm}.$$

Hence

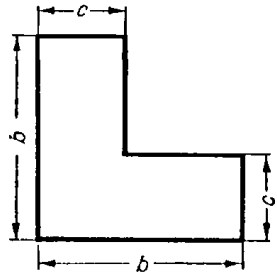
$$z_{\max} = (20 \times 0{\cdot}896) + (19 \times 0{\cdot}445) = 26{\cdot}37 \text{ cm}$$

and the section modulus

$$W_y = \frac{J_y}{z_{\max}} = \frac{70{,}400}{26{\cdot}37} = 2670 \text{ cm}^3.$$

4.44. Find the principal second moments of area of the thick-walled angle profile shown in the figure, if $b = 2c$.

Find also the product second moment of area of the cross-section about the centroidal axes parallel to the sides of the angle.



PROBLEM 4.44

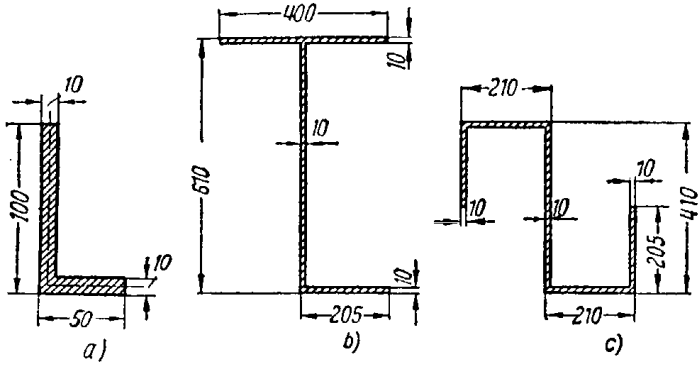
Answer: $J_{\max} = \frac{5}{64}\, b^4$; $J_{\min} = \frac{7}{192}\, b^4$;

$$J^0_{yz} = \frac{J_{\max} - J_{\min}}{2} \sin 2\beta = \frac{b}{2}\left(\frac{5}{64} - \frac{7}{192}\right) \sin(-90°) \approx 0{\cdot}021\, b^4.$$

4.45. Find the product second moment of area of an unequal-sided angle with dimensions of 100 mm × 75 mm × 10 mm (OST 10015-39) about the centroidal axes parallel to the sides.

Answer: $J^0_{yz} = -65{\cdot}7$ cm^4.

4.46. Find the position of the principal centroidal axes, and calculate the values of the principal second moments of area for the cross-sections shown in the figure. The dimensions given are in millimetres.



PROBLEM 4.46

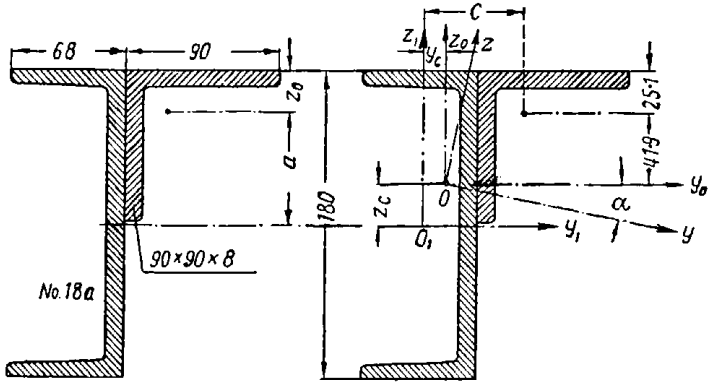
Answer:

(a) $\alpha = 14°23'$; $J_y = 150$ cm^4; $J_z = 15{\cdot}7$ cm^4;

(b) $\alpha = 6°26'$; $J_y = 69{,}700$ cm^4; $J_z = 6930$ cm^4;

(c) $\alpha = 31°15'$; $J_y = 78{,}100$ cm^4; $J_z = 6000$ cm^4.

4.47. Find the position of the principal centroidal axes, and calculate the value of the principal second moments of area for the cross-section made up of rolled sections (see figure).



PROBLEM 4.47

Solution. We write out the required data from the tables of standard sections (see Appendix A):

No. 18a channel: $F_{ch} = 25{\cdot}69$ cm^2;
$J_{max} = 1272{\cdot}7$ cm^4;
$J_{min} = 98{\cdot}6$ cm^4;
$y_0 = 1{\cdot}88$ cm;

90 mm × 90 mm × 8 mm angle: $F_{an} = 14{\cdot}0$ cm^2;
$J_y = 106$ cm^4;
$z_0 = 2{\cdot}51$ cm;
$J_{max} = 168$ cm^4;
$J_{min} = 43{\cdot}6$ cm^4.

The total area of cross-section $F = 25{\cdot}69 + 14{\cdot}0 = 39{\cdot}69$ cm^2.

We shall take the principal axes y_1 and z_2 of the channel as the auxiliary axes. The first moments of area of the cross-section about the axes y_1 and z_1 will be:

$$S'_y = F_{an}d = 14\,(9{\cdot}0 - 2{\cdot}51) = 90{\cdot}86\ \text{cm}^3;$$
$$S'_z = F_{an}c = 14\,(1{\cdot}88 + 2{\cdot}51) = 61{\cdot}46\ \text{cm}^3.$$

The distances from the axes y_1 and z_1 to the centroid of the composite cross-section is:

$$z_c = \frac{S'_y}{F} = \frac{90{\cdot}86}{39{\cdot}69} = 2{\cdot}29 \approx 2{\cdot}3\ \text{cm};$$
$$y_c = \frac{S'_z}{F} = \frac{61{\cdot}46}{39{\cdot}69} = 1{\cdot}55\ \text{cm}.$$

We calculate now the second moments of area about the centroidal axes y_0 and z_0, making use of the theorem of parallel axes: $J^0_y = J'_y + Fa^2$ and $J^0_{zy} = J'_{zy} + Fab$:

$$J^0_z = [1273 + (25{\cdot}69 \times 2{\cdot}3^2)] + [106 + (14 \times 4{\cdot}19^2)]$$
$$= 1760\ \text{cm}^4;$$
$$J^0_z = [98{\cdot}6 + (25{\cdot}69 \times 1{\cdot}55^2)] + [106 + (14 \times 2{\cdot}84^2)]$$
$$= 380\,\text{cm}^4;$$
$$J^0_{zy} = (25{\cdot}69 \times 1{\cdot}55 \times 2{\cdot}3) + \left[(14{\cdot}0 \times 2{\cdot}84 \times 4{\cdot}19) + \frac{168 - 43{\cdot}6}{2}\right] = 320\ \text{cm}^4.$$

(The last term of the formula in the brackets represents the product second moment of area of the angle about its own centroidal axes parallel to the sides, equal to

$$J^0_{yz} = \frac{J_{max} - J_{min}}{2} \sin 2\beta,$$

where $\beta = 45°$, and J_{max} and J_{min} are the principal second moments of area of the angle.)

The angle of inclination of the principal axes to the centroidal axes y_0 and z_0 are:

$$\tan 2\alpha = \frac{2J^0_{yz}}{J^0_z - J^0_y} = \frac{2 \times 320}{380 - 1760} = -0{\cdot}464;$$

$$2\alpha = -24°40'; \quad \alpha = -12°20'; \quad \sin\alpha = -0{\cdot}214;$$

$$\sin 2\alpha = -0{\cdot}417; \quad \cos\alpha = 0{\cdot}977.$$

The principal second moments of area are:

$$J_y = J^0_y \cos^2\alpha + J^0_z \sin^2\alpha - J^0_{yz} \sin 2\alpha$$

$$= (1760 \times 0{\cdot}977^2) + (380 \times 0{\cdot}214^2) + (320 \times 0{\cdot}417) = 1830\,\text{cm}^4;$$

$$J_z = J^0_y \sin^2\alpha + J^0_z \cos^2\alpha + J^0_{yz} \sin 2\alpha$$

$$= (1760 \times 0{\cdot}214^2) + (380 \times 0{\cdot}977^2) - (320 \times 0{\cdot}417) = 310\,\text{cm}^4.$$

Check.

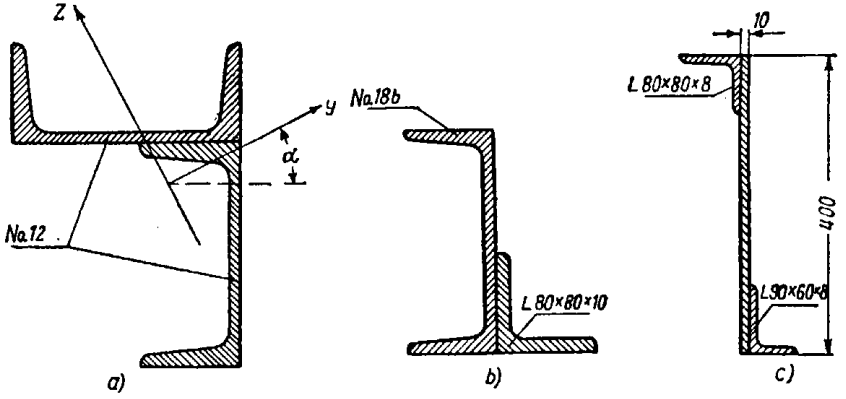
$$J_y + J_z = 1830 + 310 = 2140\,\text{cm}^4;$$

$$J^0_y + J^0_z = 1760 + 380 = 2140\,\text{cm}^4;$$

$$J_{yz} = \frac{J^0_z - J^0_z}{2} \sin 2\alpha + J^0_{yz} \cos 2\alpha$$

$$= -\left(\frac{1760 - 380}{2} \times 0{\cdot}417\right) + (320 \times 0{\cdot}909) \approx 0.$$

4.48. Find the position of the principal centroidal axes and calculate the gross principal second moments of area for the cross-sections made up of rolled profiles (see figure). The dimensions given are in millimetres.



PROBLEM 4.48

Answer:

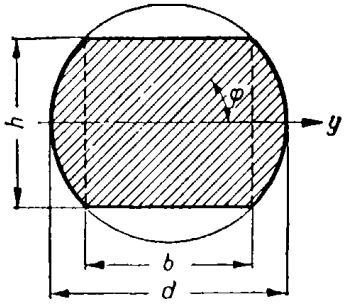
(a) $\alpha = 29°20'$; $J_y = 968\ \text{cm}^4$; $J_z = 393\ \text{cm}^4$;

(b) $\alpha = 11°40'$; $J_y = 1967\ \text{cm}^4$; $J_z = 306\ \text{cm}^4$;

(c) $\alpha = 5°19'$; $J_y \approx 11{,}170\ \text{cm}^4$; $J_z \approx 149\ \text{cm}^4$.

4.49. Find the second moment of area (about the y axis) for the cross-section formed from a circle by cutting off two segments (see figure).

Answer: $J_y = \dfrac{d^4}{32}\left(\Phi - \dfrac{\sin 4\Phi}{4}\right)$.



PROBLEM 4.49

4.50. Determine, from the viewpoint of strength in flexure, the most advantageous dimensions for a rectangular cross-section of a beam, which may be sawn out of a log having a diameter d (see figure of Problem 4.49).

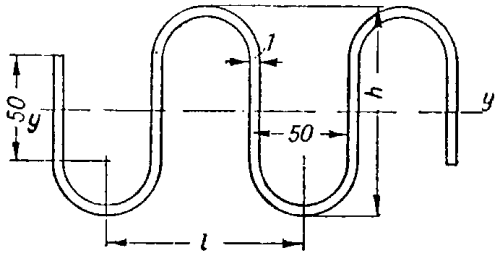
Instruction. Express one of the dimensions of the rectangle in terms of the diameter and the other dimension; then form an expression for the section modulus W_y, and investigate it for a maximum.

Answer: $W_y = \dfrac{b}{6}(d^2 - b^2)$; $b = \dfrac{d}{\sqrt{3}} \approx 0{\cdot}577d$;

$$h = \frac{d\sqrt{2}}{\sqrt{3}} \approx 0{\cdot}816d.$$

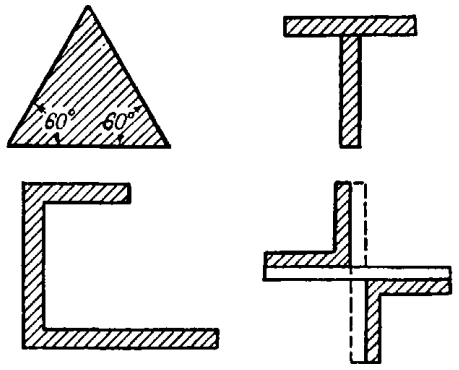
4.51. Find the second moment of area about the centroidal y axis of the cross-section of a sheet of corrugated iron with a width $a = 1{\cdot}0$ m, if the sheet thickness b is 1 mm, the height h of corrugation is 100 mm and the pitch l is 100 mm. The mean radius r of the semicircles is 25 mm (see figure).

Answer: $J_y = 280$ cm^4.



PROBLEM 4.51

4.52. Show the approximate position of the principal axes, and the ellipses of second moments of area for the figures given below.



PROBLEM 4.52

16. NORMAL STRESSES ACCOMPANYING FLEXURE

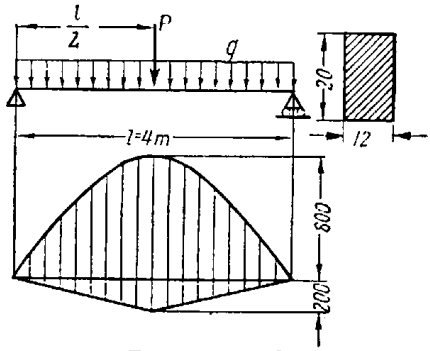
4.53. Determine the maximum normal stresses in the critical cross-section of a simply supported timber beam of span $l = 4$ m, loaded by a uniformly distributed load $q = 400$ kg/m and a concentrated force $P = 200$ kg applied at the mid-span. The cross-section of the beam is a rectangle having dimensions $b \times h = 12 \times 20$ cm^2.

Solution. The maximum bending moment, because of symmetry, will be at the mid-span, whereby its component due to the uniformly distributed load is

$$M_q = \frac{ql^2}{8} = \frac{400 \times 4^2}{8} = 800 \text{ kgm},$$

and its component due to the concentrated force is

$$M_p = \frac{pl}{4} = \frac{200 \times 4}{4} = 200 \text{ kgm}.$$



PROBLEM 4.53

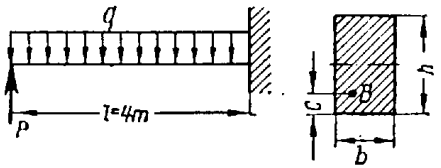
The resultant bending moment at the mid-span is $\max M = 800 + 200 = 1000 \text{ kgm} = 1 \times 10^5 \text{ kgcm}$. The bending moment diagram is shown in the figure.

The section modulus is $W = bh^2/6 = (12 \times 20^2)/6 = 800 \text{ cm}^3$. The maximum normal stress at the extremities of the critical cross-section of the beam is

$$\sigma_{\max} = \frac{\max M}{W} = \frac{10^5}{800} = 125 \text{ kg/cm}^2.$$

4.54. A wooden beam of rectangular cross-section with dimensions $20 \times 30 \text{ cm}^2$, clamped at one end in a wall, is supported at the other end by a force $P = 500$ kg, and loaded by a continuous load $q = 600$ kg/m. The cross-section of the beam is a rectangle with the sides $b = 20$ cm and $h = 30$ cm. Construct Q and M diagrams, and calculate the normal stresses in the extremities of the critical cross-section, and at a point located a distance $c = 4$ cm from the lower edge.

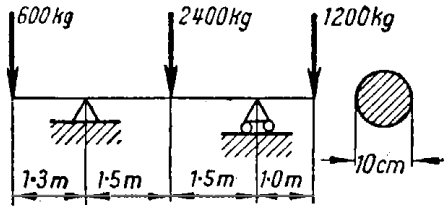
Answer: $\sigma_{\max} = \pm 93 \text{ kg/cm}^2$; $\sigma_c = -68 \text{ kg/cm}^2$.



PROBLEM 4.54

4.55. Construct M and Q diagrams, and determine the maximum normal stress for the beam of circular cross-section shown in the figure.

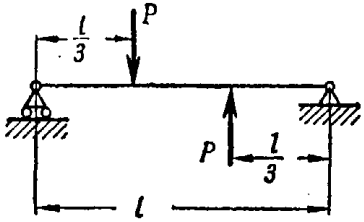
Answer: $\sigma = 1220$ kg/cm^2.



PROBLEM 4.55

4.56. Select a standard OST I-beam 6 m in span, loaded by forces P as indicated in the figure. Take the allowable stress $[\sigma]$ as 1600 kg/cm^2.

Answer: I No. 20b.



PROBLEM 4.56

4.57. *If the allowable stress $[\sigma]$ is 1600 kg/cm^2, construct shear force and bending moment diagrams, and choose cross-sections for the beams (from the standard profiles), clamped at one end and loaded as indicated in the table. The dimensions given are in metres.

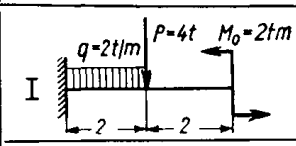
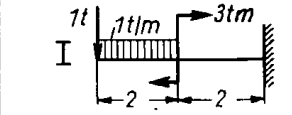
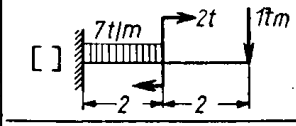
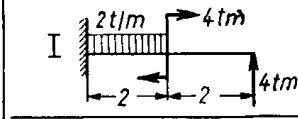
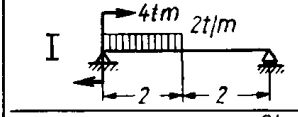
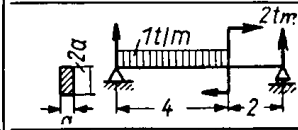
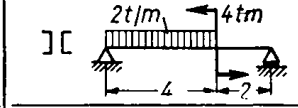
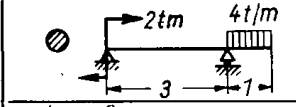
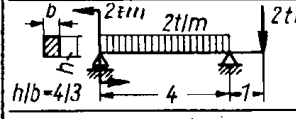
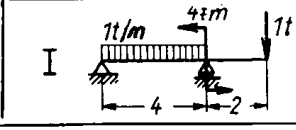
Answer: See diagrams a–d on page 183.

4.58. Construct shear force and bending moment diagrams, and select cross-sections for the two-support beams shown in the figure. Assume the following allowable stresses: $[\sigma] = 1600$ kg/cm^2 for rolled sections; $[\sigma] = 120$ kg/cm^2 for square and circular sections (pine). The lengths of beam sections are indicated in metres.

Answer: See diagrams e–j on page 183.

* In the table 10 diagrams of the same type are shown designated by letters a–d (Problem 4.57) and e–j (Problem 4.58).

TABLE TO PROBLEMS 4.57 AND 4.58

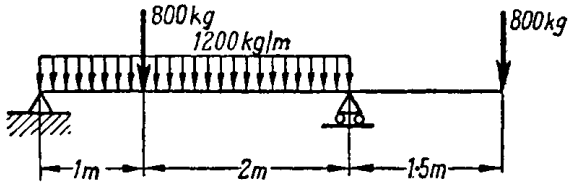
	Loading Diagram	Answer		
		M (t m)	Q (t)	Section
a		8·0	10·0	I No. 30b
b		3·0	7·0	I No. 27a
c		3·0	8·0	2 [No. 24a
d		4·0	8·0	I No. 27b
e		2·0	5·0	I No. 22b
f		2·33	3·33	a = 16 cm
g		6·0	9·0	2 [No. 24a
h		4·0	2·0	d = 25·5 cm
i		4·0	2·0	18 × 24 cm²
j		2·5	3·12	I No. 18

4.59. Two No. 20a I-beams are set side by side and support a wall of constant height. They are replaced by one I-beam. What section must be taken?

Answer: I No. 27a.

4.60. A wooden beam of rectangular cross-section 15 cm wide and 30 cm deep is loaded as shown in the figure. Construct M and Q diagrams, and determine the maximum normal stress.

Answer: $\sigma_{max} = 60$ kg/cm^2.



PROBLEM 4.60

4.61. A cast iron pipe 25 cm in external diameter, having a wall thickness of 1 cm, rests on two supports 12 m from each other. It is filled with water. What are the maximum normal stresses in the pipe, if the specific gravity of cast iron is 7·8?

Answer: $\sigma_{max} = 412$ kg/cm^2.

4.62. A beam 2 m in span rests freely on two supports. It has a rectangular cross-section 6 cm wide and 10 cm deep. It is loaded by a concentrated force of 0·5 t applied at the mid-span, and a concentrated force of 1 t applied at a distance of 0·33 m from the right-hand support. Determine the normal stress at a point in the cross-section located 0·33 m from the left-hand support. The point is located at a distance of 2 cm from the top face of the beam. The forces are directed vertically downwards.

Answer: $\sigma = -83$ kg/cm^2.

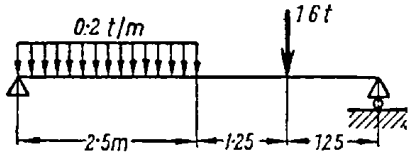
4.63. A beam 6 m long rests on two supports located at a distance of 4·5 m from each other, the right-hand end of the beam being overhung by 0·5 m. The beam weighs 66 kg/m; at a distance of 2·25 m from the left-hand support the beam is loaded by a concentrated force of 1 t. Determine the magnitude of a load which

will have to be applied at the end of the left-hand overhang for the bending moment at the point of application of the 1 t force to be zero. Determine the reactions under these conditions.

Answer: $P = 2540$ kg.

4.64. Plot bending moment and shear force diagrams for the beam shown in the figure. Find the diameter of the beam, if $[\sigma]$ is 1100 kg/cm^2.

Answer: $d = 11{\cdot}3$ cm.



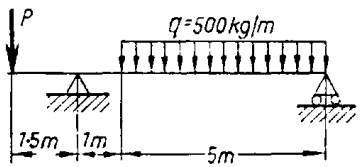
PROBLEM 4.64

4.65. Two beams, one of which is of square cross-section and the other of circular cross-section, have the same length and cross-sectional area, and are made of the same material. The safe value of a concentrated load, applied at the mid-span of the beam of the circular cross-section, is 1 t. What will be the value of a safe load, applied also at the mid-span, for the beam of square cross-section and of the same span?

Answer: 1180 kg.

4.66. Find the maximum value of load P which may be applied safely to the beam, if the allowable stress is 1600 kg/cm^2 and the cross-section of the beam is I No. 30 a. The diagram of the beam is shown in the figure.

Answer: $P = 6{\cdot}37$ t.



PROBLEM 4.66

4.67. The value of a concentrated load which can safely be supported at the mid-span of a simply supported beam of uniform cross-section is 5 t. Determine the maximum value of a load which may be applied to this beam: (a) at the quarter-span, and (b) uniformly distributed over its whole length.

Answer: (a) 6.67 t; (b) 10 t.

4.68. Two workers have to carry a steel ($\gamma = 7{\cdot}85$ g/cm^3) rod 12 m long having a cross-section of 3×3 cm^2. What will be the maximum normal stress, if they hold it by the ends? At what places (at equal distances from the ends) must they support it for the normal stress not to exceed 1400 kg/cm^2?

Answer: (1) $\sigma = 2820$ kg/cm^2; (2) $a \approx 4{\cdot}25$ m.

4.69. Beams 10×30 cm in cross-section and 8·5 m long are used in a building. A question has arisen whether it is possible to use one of these beams as a temporary bridge, for workmen passing from one part of the building to another. The span is 7 m. Is such an application of the beam permissible if it is laid on its longer side? If this is the case, may more than one workman use it at the same time? The allowable stress is 100 kg/cm^2; the weight of a workman is 80 kg.

Answer: Three workmen may use it at the same time.

4.70. The cross-section of a cast-iron beam having a span $l = 4$ m is shown in the figure. Determine the value of an allowable load which can be applied at the mid-span, if the allowable stress $[\sigma]$ is 150 kg/cm^2 in tension. What value will the maximum compressive stress then be?

Solution. Dividing the cross-section into three rectangles we find that the cross-sectional area F is 225 cm^2; the first moment of area of the cross-section about the y_1 axis passing through the lower edge of the figure is

$$S'_y = [5 \times 10\ (30 - 2{\cdot}5)] + [2{\cdot}5 \times 20\ (10 + 5)] + (5 \times 25 \times 2{\cdot}5) = 2437{\cdot}5 \text{ cm}^3;$$

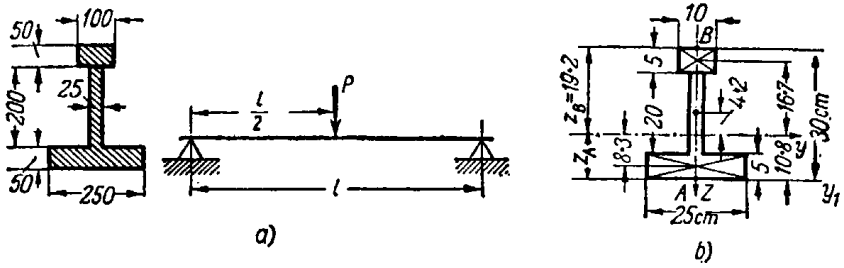
the distance from the y axis to the centroid of the cross-section is

$$z_c = \frac{S'_y}{F} = \frac{2437{\cdot}5}{225} = 10{\cdot}8 \text{ cm}.$$

The second moment of area of the cross-section about the centroidal axis Oy is calculated by the formula of parallel axes:

$$J_y = \frac{10 \times 5^3}{12} + (10 \times 5 \times 16{\cdot}7^2) + \frac{2{\cdot}5 \times 20^3}{12} + (2{\cdot}5 \times 20 \times 4{\cdot}2^2) + \frac{25 \times 5^3}{12} + (5 \times 25 \times 8{\cdot}3^2)$$

$$= 25{,}470 \text{ cm}^4.$$



PROBLEM 4.70

The value of the bending moment which can safely be allowed for the beam is determined by the formula

$$\max \sigma = \frac{\max M}{J_y} \times z_A \leq [\sigma]$$

where z_A is the distance from the neutral axis to the extreme fibre in tension, equal to 10·8 cm.

Consequently

$$[M] = \frac{150 \times 25{,}470}{10{\cdot}8} = 354{,}000 \text{ kg cm} = 3{\cdot}54 \text{ t m}.$$

According to the loading diagram the maximum bending moment will be

$$\max M = \frac{Pl}{4} \leq [M].$$

Hence the required value of the force is

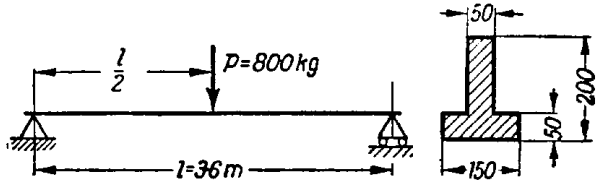
$$P = \frac{4[M]}{l} = \frac{4 \times 3{\cdot}54}{2{\cdot}4} = 5{\cdot}9 \text{ t}.$$

The maximum compressive stress in the critical section will be

$$\max \sigma = \frac{\max M z_B}{J_y} = \frac{354000 \times 19{\cdot}2}{25{,}470} = 267 \text{ kg/cm}^2.$$

4.71. A T-beam (see figure) is loaded by a concentrated force at the mid-span. The material of the beam has the limit of proportionality 300 kg/cm² in tension, and 600 kg/cm² in compression. Determine the maximum tensile and compressive stresses at the mid-span of the beam, and find the factor of safety with respect to the limit of proportionality.

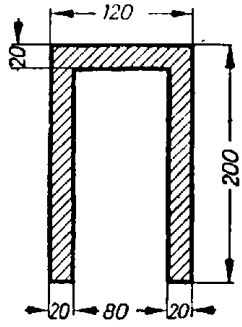
Answer: $\sigma_{max} = 102$ kg/cm²; $\sigma_{min} = -170$ kg/cm².



PROBLEM 4.71

4.72. Determine the value of the maximum normal stresses in a beam of Π—shaped cross-section with a span $l = 1{\cdot}1$ m, clamped at one end and loaded by a force $P = 4{\cdot}0$ t at the other end. The dimensions of the cross-section given in the figure are in millimetres.

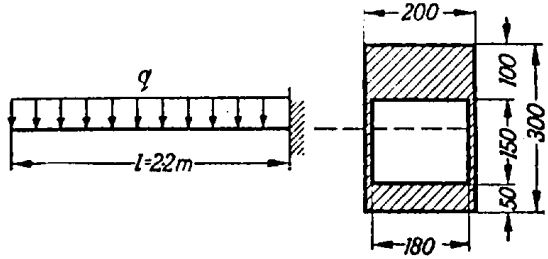
Answer: $\sigma_{max} = 1350$ kg/cm².



PROBLEM 4.72

4.73. Determine the load-carrying capacity of a cast iron beam the cross-section of which, together with the loading diagram, is shown in the figure. The allowable stress is 120 kg/cm² in tension, and 500 kg/cm² in compression.

Answer: $q = 1{\cdot}41$ t/m.



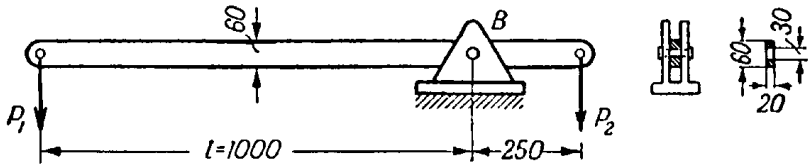
PROBLEM 4.73

4.74. How will the load-carrying capacity of a beam of square cross-section change if it is turned through an angle of 45° about the longitudinal axis?

Answer: It is reduced by 29·3%.

4.75. A steel lever of a braking mechanism consists of a bar of rectangular cross-section of 20×60 mm², being supported at point B (see figure) by a bolt passing through the hole in the bar. The centre of the hole is located on the axis of the bar; its diameter d is 30 mm. Find what forces may be applied safely to the ends of the lever if the allowable stress $[\sigma]$ is 1400 kg/cm². Take the weakening effect of the bolt hole in the cross-section into account.

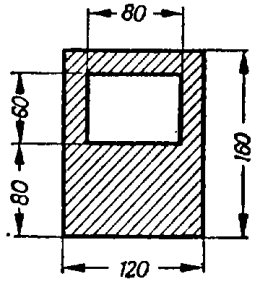
Answer: $P_1 = 147$ kg; $P_2 = 588$ kg.



PROBLEM 4.75

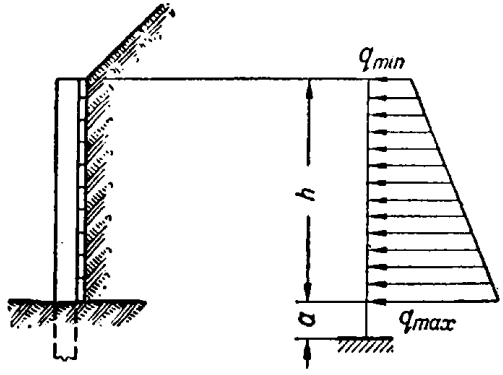
4.76. A simply supported cast iron beam of hollow cross-section carries a continuous uniformly distributed load over the whole span $l = 2$ m. The dimensions of the cross-section shown in the figure are in mm. Find the value of the load intensity q, if it causes a compressive stress σ of 400 kg/cm^2 in the critical section of the beam.

Answer: $q = 3$ t/m.



PROBLEM 4.76

4.77. A retaining wall for an embankment consists of round piles arranged at a distance of 1·2 m from each other, and cover planks having a width b of 20 cm (see figure). The pressure from the side of the embankment increases linearly from $q_{\min} = 0{\cdot}5$ t/m^2 to $q_{\max} = 1{\cdot}8$ t/m^2. The height h of the retaining wall is 1·5 m. Due to yielding of the ground at the free surface, the fixing point of the piles is assumed to be below the ground surface by an amount a,



PROBLEM 4.77

which in the given case may be taken as 0·25 m. Considering the cover planks as beams hinged at the piles and having a span l of 1·2 m, determine their required thickness t, and choose the diameter d of the piles. The allowable stress $[\sigma]$ for the timber in flexure is 100 kg/cm² in tension and in compression.

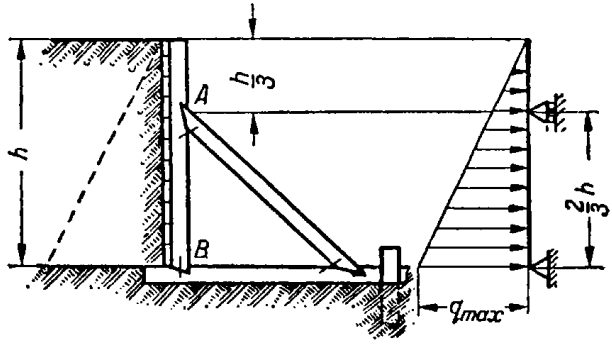
Instruction. The intensity of the load on the piles varies from $q_1 = q_{\min} \times l = 0{\cdot}6$ t/m to $q_2 = 2{\cdot}16$ t/m. The pressure q on the bottom plank is $(b/100)q_{\max}$.

Answer: For piles: $\max M = 1{\cdot}78$ tm; $d = 26$ cm. For the planks: $\max M = 0{\cdot}065$ tm; $t = 4{\cdot}4$ cm.

4.78. Choose a square cross-section for the uprights, and a thickness for the planking of the retaining wall shown in the figure. The uprights, set at every 2 m, rest on foundation beams and are supported by struts; as a result, each upright may be considered as a beam hinged at points A and B, and loaded by the pressure transmitted from the earth through the planking. Neglect the longitudinal force set up in the upright.

The planks having a width $b = 20$ cm must be considered as hinged beams cut through at the supports. The height h of the embankment is 1·5 m. The earth pressure increases linearly from zero to $q_{\max} = 1{\cdot}3$ t/m. The allowable stress $[\sigma]$ in flexure is 100 kg/cm² in tension and in compression.

Answer: The cross-section for the uprights is 16×16 cm²; for the planking $t = 6{\cdot}25$ cm.



PROBLEM 4.78

17. SHEAR STRESSES ACCOMPANYING FLEXURE

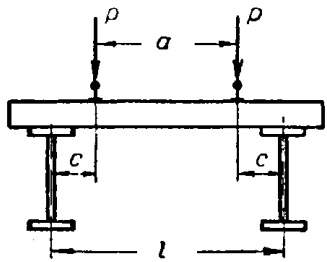
4.79. A beam of rectangular cross-section, 7·5 cm wide and 15 cm deep, with a span of 2 m, is simply supported. Determine the normal and shear stresses at a point 5 cm above the neutral axis in a cross-section located at a distance of 75 cm from the left-hand support. The beam is loaded by a concentrated force of 400 kg applied at the mid-span.

Answer: $\sigma = 35{\cdot}5\ \text{kg/cm}^2$; $\tau_c = 1{\cdot}5\ \text{kg/cm}^2$.

4.80. Choose the dimensions of the rectangular cross-section of a wooden bridge beam, with the ratio of sides $h/b = 4/3$. The allowable bending stress $[\sigma]$ is 100 kg/cm² in tension and in compression; the shear stress $[\tau]$ is 25 kg/cm². The span l of the beam is 2·0 m, the distance a between the axes of the rails is 1·6 m. The thrust P transmitted to the beam from the wheels of rolling stock, via the rails, is 9·8 t (see figure).

Instruction. A preliminary choice of the cross-section is carried out for the limiting strength condition dependant on normal stresses in the cross-section where $M = M_{\max}$. Then the adequacy of this cross-section is checked for shear stresses and, if necessary, the final selection of the cross-section is carried out for the limiting strength condition dependant on the shear stresses.

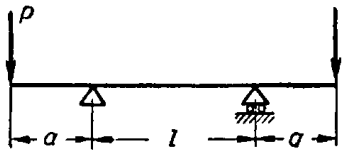
Answer: 21 × 28 cm².



PROBLEM 4.80

4.81. Select a cross-section for the I-beam shown in the figure, if the allowable tensile and compressive bending stresses $[\sigma]$ are 1600 kg/cm², and the shear stress $[\tau]$ is 1000 kg/cm². The beam is bent by concentrated forces $P = 20$ t. $l = 2$ m; $a = 0{\cdot}2$ m.

Answer: I No. 24b.



PROBLEM 4.81

4.82. A beam of rectangular cross-section, 10 cm wide and 15 cm deep, is simply supported and has a span of 4 m. It is loaded over a length of 2 m, symmetrically about the mid-span, by a uniformly distributed load having an intensity 2 t/m. Determine the magnitude of the shear stress at a point 4 cm from the bottom face of the beam. The cross-section is located 0·5 m from the right-hand support.

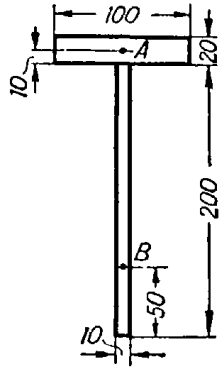
Answer: $\tau = 15{\cdot}6$ kg/cm^2.

4.83. Determine the maximum shear stresses in a simply supported No. 18 I-beam with a span of 4 m. The beam is loaded over the whole span by a uniformly distributed load giving rise to a maximum normal stress of 1400 kg/cm^2.

Answer: max $\tau = 258$ kg/cm^2.

4.84. The cross-section of a beam is shown in the figure. Determine the shear stresses at points A and B due to the action of a shear force $Q = 6$ t.

Answer: $\tau_A = 19$ kg/cm^2; $\tau_B = 207$ kg/cm^2.

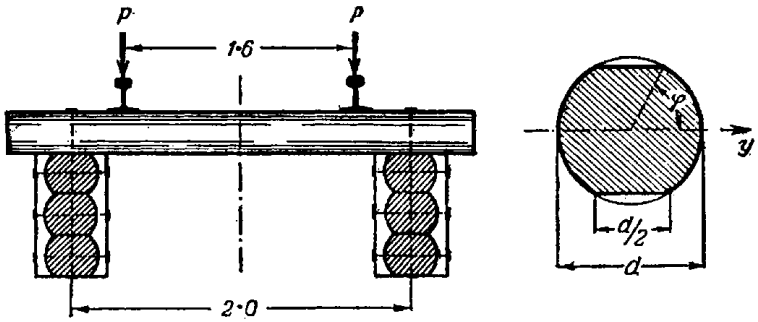


PROBLEM 4.84

4.85. Determine the value of the maximum shear stresses in a pipe subjected to flexure, with an external diameter $D = 20$ cm, and an internal diameter $d = 18$ cm. The shear force Q is 6 t.

Answer: $\max \tau = 200$ kg/cm^2.

4.86. Transverse bridge beams, in the form of logs having a diameter $d = 25$ cm, and cut away at both sides, carry a load $P = 6$ t transmitted through the rails from the wheels of a locomotive (see figure). Check the adequacy of the beams, if the allowable bending stress $[\sigma]$ is 100 kg/cm^2, and the allowable (bending) shear stress $[\tau]$ is 20 kg/cm^2.

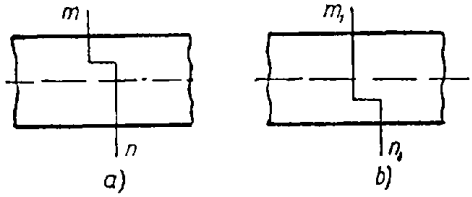


PROBLEM 4.86

Instruction. The second moment of area of the cross-section of a circle with segments cut off is $J_y = d^4/32\,(\Phi - 0{\cdot}25 \sin 4\Phi)$. The first moment of area of one half of the cross-section about the centroidal y axis is $S_y = d^3/12\,(1 - \cos^3 \Phi)$. (See Problem 4.49.)

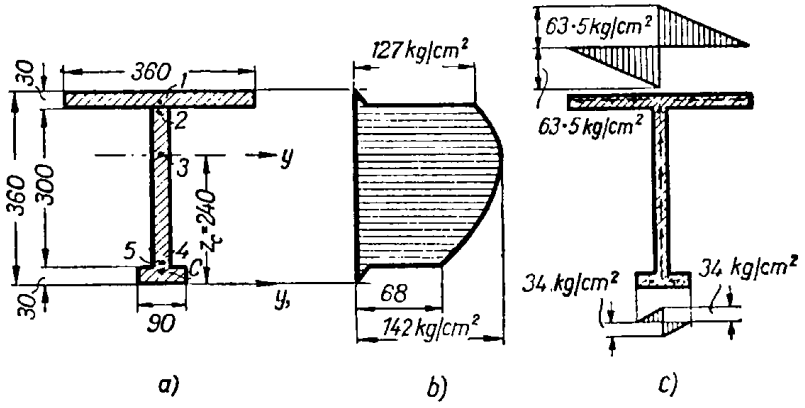
Answer: $\max \sigma = 86$ kg/cm^2; $\max \tau = 18$ kg/cm^2.

4.87. Indicate the direction of the shear stresses which are transmitted from the right-hand part to the left-hand one, and vice versa, through the stepped section of a beam of rectangular cross-section, if the shear force in the section is positive, and if the shear force in the section is negative (see figure).



PROBLEM 4.87

4.88. Plot a diagram showing the variation of the shear stress along the depth of the non-symmetrical I-section shown in the figure. Calculate the values of τ at the points where the web joins the flanges, and at the neutral layer, if $Q = 12$ t. The dimensions of the cross-section are given in millimetres.



PROBLEM 4.88

Solution. The distance z_c from the y axis to the centroid of the cross-section is calculated from the formula $z_c = S_y/F$, and is approximately 24 cm.

The second moment of area J_y of the cross-section about the neutral axis y is 35,640 cm^4.

The first moment of area of the parts of the cross-section in shear are respectively:

top flange: $S_1 = 3 \times 36 \times 10{\cdot}5 = 1135$ cm^3;

bottom flange: $S_2 = 9 \times 3 \times 22{\cdot}5 = 607$ cm^3;

one half cross-section:

$$S = S_2 + \left(3 \times 21 \times \frac{21}{2}\right) = 607 + 663 = 1270\,\text{cm}^3.$$

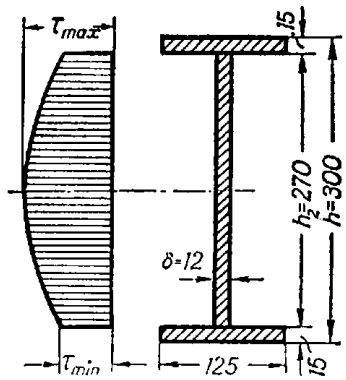
When calculating the horizontal shear stresses in the flanges, $0{\cdot}5S_1$ and $0{\cdot}5S_2$ are introduced respectively for points 1 and 4 in the τ formula, and the thickness of the flange is taken as the width of the cross-section.

The calculation of the shear stresses at the required points is carried out in the tabular form:

Point	Location of point	Width of section $b(z)$, cm	First moment of area $S_y(z)$, cm³	$\tau = \frac{QS(z)}{J_y b(z)}$, kg cm²
1	Top flange	3·0	567·5	$\tau_1 = \frac{12{,}000 \times 567{\cdot}5}{35{,}640 \times 3} = 63{\cdot}5$
2	Web	3·0	1135	$\tau_2 = \frac{12{,}000 \times 1135}{35{,}640 \times 3} = 127$
3	Web	3·0	1270	$\tau_3 = \frac{12{,}000 \times 1270}{35{,}640 \times 3} = 142$
4	Web	3·0	607	$\tau_4 = \frac{12{,}000 \times 607}{35{,}640 \times 3} = 68$
5	Bottom flange	3·0	303·5	$\tau_5 = \frac{12{,}000 \times 303{\cdot}5}{35{,}640 \times 3} = 34$

In the web the shear stresses are directed vertically, whereas in the flanges they are directed horizontally, and in opposition to each other. The diagrams showing the variation in the shear stresses over the cross-section, together with their direction, are given in figures (b) and (c).

4.89. Find the values of the maximum, and the minimum shear stresses in the web of an I-beam, due to the shear force $Q = 30$ t.



PROBLEM 4.89

Find the proportion of the shear force, which is carried by the web. The dimensions of the cross-section of the beam shown in the figure are in millimetres.

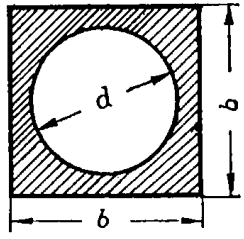
Answer: $\tau_{max} = 980\ \text{kg/cm}^2$; $\tau_{min} = 698\ \text{kg/cm}^2$;
$Q_{web} = 0{\cdot}96\ Q \approx 28{\cdot}8\ \text{t}$.

4.90. A simply supported wooden beam of hollow square cross-section, with a span $l = 3\ \text{m}$, is loaded by a continuous load $q = 1\ \text{t/m}$. The dimensions of the cross-section are $20 \times 20\ \text{cm}^2$ outside, and $10 \times 10\ \text{cm}^2$ inside. Find the maximum values of the normal stress and the shear stress in the sections, where the bending moment and the shear force are M_{max} and Q_{max} respectively.

Answer: $\max \sigma = 90\ \text{kg/cm}^2$; $\max \tau = 10{\cdot}5\ \text{kg/cm}^2$.

4.91. A simply supported cast iron beam of hollow cross-section is loaded by two equal, symmetrically placed loads $P = 6{\cdot}0\ \text{t}$, applied at a distance $a = 0{\cdot}7\ \text{m}$ from the respective supports (see figure). Find the maximum values of the normal stress and the shear stress in the critical section of the beam, if the span l is 3·0 m, and the dimensions of the cross-section are: $b = 24\ \text{cm}$ and $d = 20\ \text{cm}$.

Answer: $\max \sigma = 254\ \text{kg/cm}^2$; $\max \tau = 80\ \text{kg/cm}^2$.

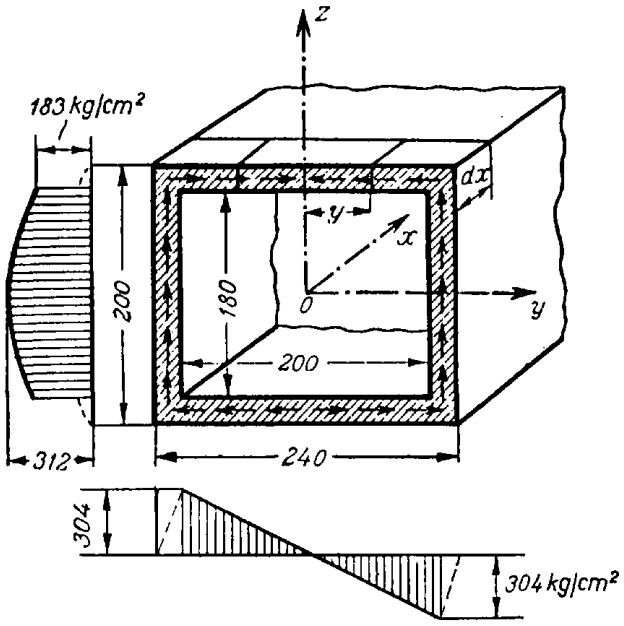


PROBLEM 4.91

4.92. A steel beam of rectangular hollow cross-section, with a span $l = 4\ \text{m}$, is simply supported, and loaded by two equal forces P applied at equal distances $a = 0{\cdot}5\ \text{m}$ from the respective supports. The cross-section of the beam is shown in the figure; the dimensions are given in millimetres. Find the magnitude of the forces P from the limiting strength condition dependent on normal stresses,

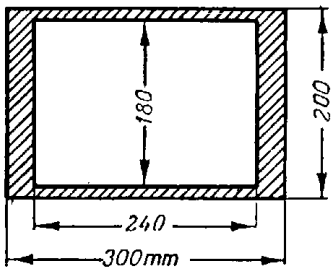
if $[\sigma]$ is 1600 kg/cm². Plot diagrams showing the variations in the shear stress along the depth and along the width of the cross-section.

Answer: $P = 20$ t. The diagrams of $\tau(y)$ and $\tau(z)$ are shown in the figure. At the places where the vertical walls meet the horizontal walls, they are given conditionally.



PROBLEM 4.92

4.93. Determine the value of the maximum shear stress occurring in the walls of a beam of hollow rectangular cross-section (see figure), subjected to a vertical shear force $Q = 24$ t. In which



PROBLEM 4.93

walls will the shear stress be greater—in the vertical, or in the horizontal walls? Plot diagrams showing the distribution of shear stress along the depth and along the width of the cross-section.

Answer: $\max \tau = 395$ kg/cm^2. The value of the maximum shear stress in the horizontal walls is 30% higher than that at the neutral layer in the vertical walls.

4.94. How will the ratio of the values of the horizontal and vertical shear stresses in the cross-section of the beam considered in the preceding problem change, if the cross-section is turned through 90°, thus making the long side its depth?

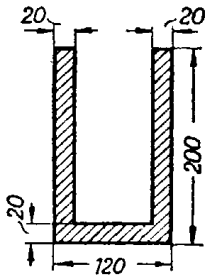
Answer: The maximum shear stresses in the vertical walls of the cross-section will become 3·9 times greater than those in the horizontal walls.

4.95. A simply supported beam of rectangular cross-section, with a span $l = 4$ m, is loaded by a continuous uniformly distributed load $q = 4$ t/m. Find the value of the maximum shear stress in the cross-section at the mid-span of the beam, if the dimensions of the cross-section are 10×20 cm^2. In what plane will τ_{max} occur?

Answer: $\tau_{max} = 600$ kg/cm^2.

4.96. Find the maximum values of the normal stress and the shear stress in a simply supported beam of box section (see figure), loaded by two concentrated forces 15 t each. The span of the beam is 3 m. The forces are applied at equal distances of 0·3 m from the respective supports. Indicate the distribution of the shear stress along the depth and along the width of the cross-section.

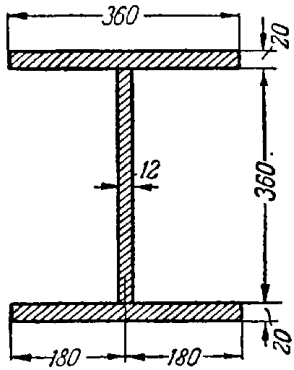
Answer: $\sigma_{max} = 1380$ kg/cm^2; $\tau_{max} = 265$ kg/cm^2.



PROBLEM 4.96

4.97. From the limiting strength condition dependent on normal stress, determine the load-carrying capacity of a wide-flanged simply supported I-beam, with a span $l = 3$ m, loaded by a concentrated force P applied at the mid-span. Calculate the value of the maximum shear stress, and plot diagrams showing the shear stress distribution along the depth of the web, and along the width of the flanges. Assume that the allowable stresses are $[\sigma] = 1600$ kg/cm^2, $[\tau] = 1000$ kg/cm^2. The dimensions of the cross-section in the figure are given in mm.

Answer: $P = 61$ t; $[\tau] = 700$ kg/cm^2.



PROBLEM 4.97

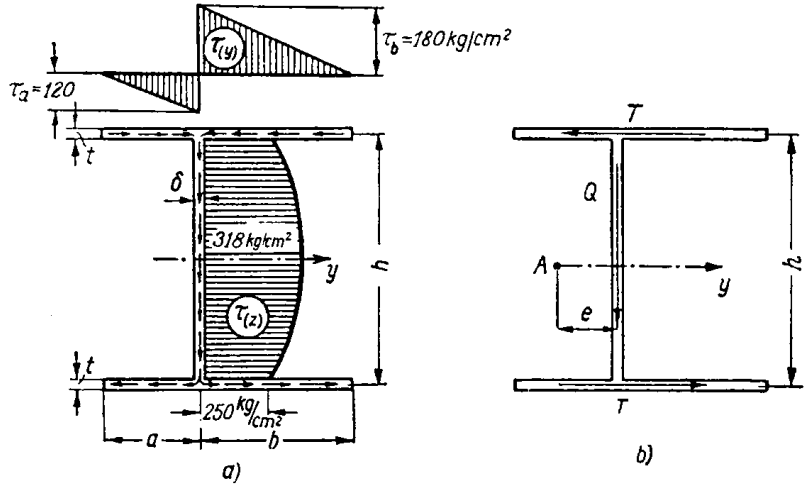
4.98. Plot diagrams showing the distribution of shear stresses along the depth of the web and along the width of the flanges, and determine the position of the centre of shear for the non-symmetrical thin-walled I-beam (see figure). The dimensions of the cross-section are: $h = 100$ mm, $a = 40$ mm, $b = 60$ mm, $t = 5$ mm, $\delta = 6$ mm. The shear force Q applied at the centre of shear is 1800 kg.

Solution. The second moment of area of the cross-section about the centroidal y axis is

$$J_y = 2(a + b)\,t\left(\frac{h}{2}\right)^2 + \frac{\delta(h - t)^3}{12} = (2 \times 10 \times 0{\cdot}5 \times 5^2) + \frac{0{\cdot}6 \times 9{\cdot}5^3}{12} = 300 \text{ cm}^4.$$

The maximum horizontal stresses in the flanges, at points where the flanges meet the web, are respectively

$$\tau_a = \frac{Qah}{2J_y} \quad \text{and} \quad \tau_b = \frac{Qbh}{2J_y}.$$



PROBLEM 4.98

The shear stress diagrams are shown in the figure. These stresses form a shear flow directed along the web, and along the flanges of the cross-section. The resultant of the vertical forces is taken to be equal to Q. The horizontal forces in the flanges are

$$T_a = \frac{1}{2}\,\tau_a at = \frac{Qa^2ht}{4J_y};$$

$$T_b = \frac{1}{2}\,\tau_b bt = \frac{Qb^2th}{4J_y}.$$

Thus, the force on each flange is $T = T_b - T_a = (b^2 - a^2)\,Qht/4\,J_y$, forming a couple with a lever arm h (figure (b)).

We shall find the position of the shear centre from the relationship

$$Th - Qe = 0.$$

Hence

$$e = \frac{Th}{Q} = \frac{h^2 t}{4J_y}(b^2 - a^2).$$

Inserting the numerical values, we have

$$e = \frac{10^2 \times 0{\cdot}5}{4 \times 300}(6^2 - 4^2) = 0{\cdot}833 \text{ cm}.$$

Since the y axis is the axis of symmetry, the shear centre A lies on this axis, at a distance $e = 0{\cdot}833$ cm from the centre line of the vertical web (see Problem 6.113, page 380).

4.99. A beam of No. 20a channel section is clamped at one end and loaded by a concentrated force $P = 2{\cdot}0$ t applied at the other end.

At what distance from the centroid of the cross-section of the beam must the force P be applied in order to avoid twisting the beam? What maximum normal stresses and maximum shear stresses will occur due to this load in the critical cross-section of the beam, if its span l is 1·2 m. The cross-section is to be considered as being made up of rectangles.

Answer: The force must be applied at the shear centre lying at a distance $y_A = -3{\cdot}8$ cm from the centroid of the cross-section; $\max \sigma = 1350$ kg/cm^2; $\max \tau = 174$ kg/cm^2.

18. COMPLETE STRENGTH-CHECK FOR BEAMS

4.100. Carry out a check on the strength of a No. 24b I-beam for normal stress, shear stress, and principal stress (in accordance with the theory of shear strain energy). The span l of the beam is 2 m. The beam is loaded by a continuous uniformly distributed load $q = 1$ t/m, and two concentrated forces $P = 20$ t applied at a distance $a = 0{\cdot}2$ m from the respective supports (see figure).

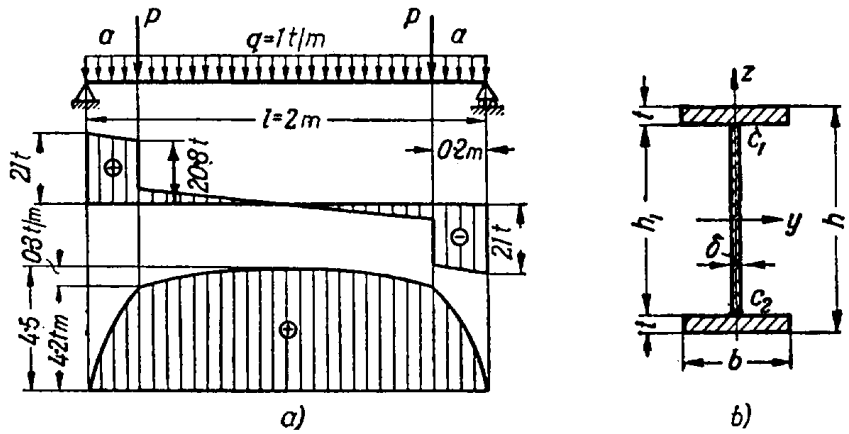
Take the allowable stress $[\sigma]$ as 1600 kg/cm^2 in tension and in compression, and $[\tau]$ as 1050 kg/cm^2 in shear. Sketch the cross-section of the beam assuming it to be made up of rectangles.

Solution. The Q and M diagrams are shown in figure (a); $Q_{\max} = 21$ t; $M_{\max} = 4{\cdot}5$ tm. The geometrical characteristics and the dimensions of the cross-section according to the table of standard sections (see Appendix A, Table 3) are respectively

$$J_y = 4800 \text{ cm}^4; \quad W_y = 400 \text{ cm}^3; \quad \frac{J_y}{S_{\max}} = 20{\cdot}4 \text{ cm};$$

$$b = 11{\cdot}8 \text{ cm}; \quad h_1 = 21.4 \text{ cm}; \quad h = 24 \text{ cm};$$

$$t = 1{\cdot}3 \text{ cm}; \quad \delta = 1{\cdot}0 \text{ cm (figure (b))}.$$



PROBLEM 4.100

(a) The check for normal stresses is carried out for a cross-section where $M = M_{\max}$, for the points furthermost from the neutral axis:

$$\max \sigma = \frac{\max M}{W_y} = \frac{450{,}000}{400} = 1125 \text{ kg/cm}^2 < 1600 \text{ kg/cm}^2.$$

With respect to σ, the cross-section has an excess $\Delta k = 100\,(1600 - 1125)/1600 \approx 36\%$ in the factor of safety. However, it is still premature to reduce the cross-section, since the two other checks have not been carried out.

(b) The check on the strength of the cross-section for shear stresses is carried out for the cross-section of the beam at the support, where $Q = 21$ t, for the points lying on the neutral axis. According to the formula of Zhuravskii

$$\tau_{\max} = \frac{Q_{\max} S_{\max}}{J_y \delta} = \frac{21{,}000}{20{\cdot}4 \times 1{\cdot}0} = 1030 \text{ kg/cm}^2 < 1050 \text{ kg/cm}^3.$$

With respect to τ, the cross-section has an excess of only 2% in the factor of safety.

(c) The check for principal stresses. Since the magnitude of the principal stresses σ_1 and σ_2 depends on both the normal stresses and the shear stresses, then the check on the strength must be carried out for that section of the beam where M and Q are both sufficiently large. Therefore we carry out the check for the section at the point of application of force P, where $Q = 20{\cdot}8$ t, and $M = 4{\cdot}2$ tm. The critical points in the chosen cross-section will be the points c_1 and c_2 ($z_c = h_1/2 = 10{\cdot}7$ cm) where the web of the beam joins the flanges. At these points the normal stresses and the shear stresses have values which are close to the maximum values in the cross-section.

The horizontal shear stresses in the thick flanges are very much lower than the stresses at points c_1 and c_2; combined with the normal stresses they only increase the value of σ_c very slightly. Therefore the check on the strength for principal stresses is superfluous for the flanges.

The condition of limiting strength, in accordance with the theory of shear strain energy in a case of plane stress has the form

$$\sigma_r = \sqrt{\sigma^2 + 3\tau^2} \leq [\sigma].$$

To determine τ we calculate the first moment of area of the flange lying above point c_1:

$$S_y(z) = b \times t \left(\frac{h_1 + t}{2}\right) = 11{\cdot}8 \times 1{\cdot}3 \left(\frac{21{\cdot}4 + 1{\cdot}3}{2}\right) = 175 \text{ cm}^3.$$

Determining the quantities for the formula of the effective stress σ_r, and then σ_r itself, we have

$$\sigma = \sigma_c = \frac{Mz_c}{J_y} = \frac{42 \times 10^5 \times 10{\cdot}7}{4800} \approx 935 \text{ cm}^2;$$

$$\tau = \tau_c = \frac{QS(z)}{J_y \delta} = \frac{20{,}800 \times 175}{4800 \times 1} \approx 760 \text{ cm}^2;$$

$$\sigma_r = \sqrt{\sigma_c^2 + 3\tau_c^2} = \sqrt{935^2 + (3 \times 760^2)} \approx$$

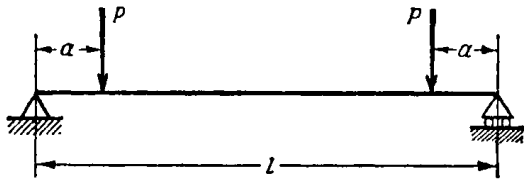
$$\approx 1625 \text{ kg/cm}^2 > 1600 \text{ kg/cm}^2.$$

The overstress is not large, constituting $\Delta k = 100 \times (1625 - 1600)/1600 = 1{\cdot}5\%$ of the total which is fully permissible.

The check carried out shows that the cross-section of the beam chosen is just adequate, although the value of the maximum normal stress in the extreme fibres is relatively low.

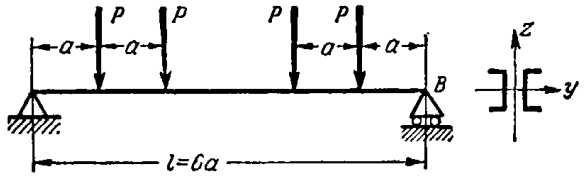
4.101. Choose an I-section for the beam (see figure), and carry out a complete check on its strength, if $P = 20$ t, $a = 0{\cdot}3$ m, $l = 3{\cdot}0$ m, $[\sigma] = 1400$ kg/cm^2, $[\tau] = 800$ kg/cm^2.

Answer: I No. 27 b.



PROBLEM 4.101

4.102. A simply supported beam with a span $l = 6$ m is loaded by four equal forces $P = 14$ t located symmetrically (see figure). Choose a cross-section for the beam, made up of two channels, satisfying the conditions of limiting strength for normal stress, shear stress and principal stress. The distance a between the forces



PROBLEM 4.102

is 0·5 m, and the allowable stresses $[\sigma]$ and $[\tau]$ are 1600 kg/cm² and 1000 kg/cm² respectively. Make use of the third theory of strength.

Answer: 2 [No. 36a.

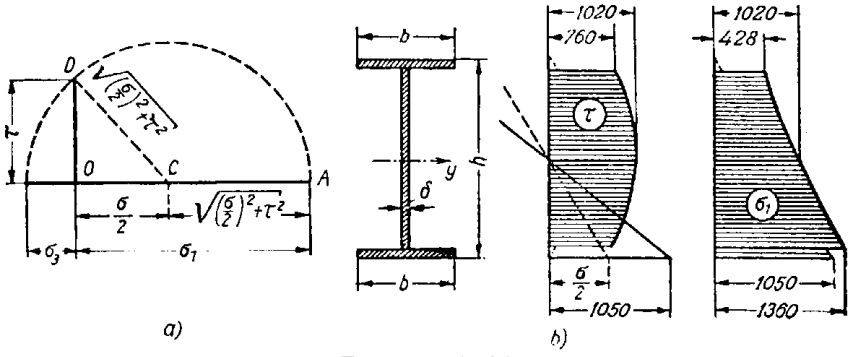
4.103. A simply supported I-beam is loaded by a force P applied at the mid-span. At what span l will the value of the maximum normal stress $\sigma_{max} = M_{max}/W$ in the critical section be equal to the value of the principal stress (for the same section) at the point where the web contacts the flange? The cross-section of the beam is I No. 24b.

Answer: $l = \dfrac{4S(z)}{\delta\sqrt{2ht}} = 98{\cdot}5$ cm.

4.104. Draw diagrams showing the distribution of normal stress, shear stress and principal stress over the depth of the cross-section in Problem 4.100, using the data given.

Instruction. Since $\sigma_1 = \sigma/2 + \sqrt{(\sigma/2)^2 + \tau^2}$, then after having drawn the σ and τ diagrams it is possible to obtain for each point the value of σ_1, considering it as a sum of the base and the hypotenuse of a right-angled triangle ODC, the base and the height of which are respectively $\sigma/2$ and τ. Rotating the hypotenuse CD into the position CA we shall obtain σ_1. Since within the limits of the flanges the τ diagram has an undefined outline, then the diagram of the principal stresses for this part of the cross-section must also be taken as undefined. The diagram of σ_3 is plotted precisely in the same manner.

Answer: See the figure.



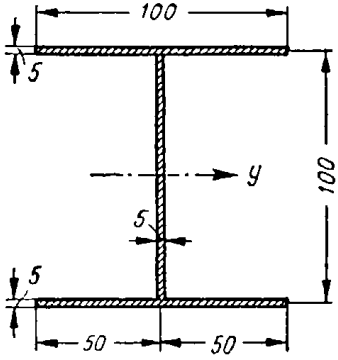
a) b)

PROBLEM 4.104

4.105. Applying the third theory of strength, carry out a complete strength-check for a beam of hollow rectangular cross-section having external dimensions of 24 × 20 cm², and internal dimensions of 20 × 18 cm² (see figure to Problem 4.92), if the bending moment M at the critical section of the beam is 9·5 tm, and the shear force Q is 20 t. At what points of the cross-section do the stresses reach their maximum values? What are the values of the principal stresses at these points?

Answer: At the points in the horizontal wall where it joins the vertical walls. $\sigma_1 = 1580$ kg/cm²; $\sigma_3 = -60$ kg/cm².

4.106. A thin-walled I-beam is subjected to flexure in the vertical plane. Determine the value of the principal stresses at the critical points in the web, and in the flange, if the bending moment M in the cross-section is 800 kgm, and the shear force Q is 3600 kg. The dimensions shown in the figure are in millimetres.

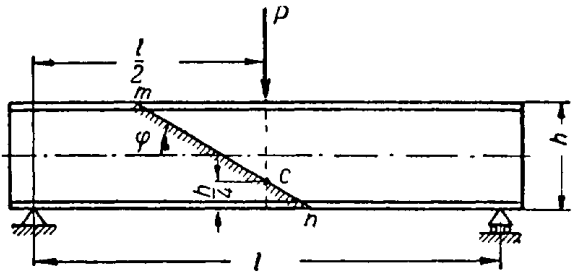


PROBLEM 4.106

Answer: In the flange $\sigma_1 = 1515$ kg/cm²; $\sigma_3 = -45$ kg/cm². In the web $\sigma_1 = 1580$ kg/cm²; $\sigma_3 = -250$ kg/cm².

4.107. A simply supported I-beam is bent by a concentrated force $P = 14$ t, applied at the middle of the span $l = 4$ m (see figure). Determine the value of the normal stress and the shear stress in a plane inclined at an angle $\Phi = 30°$ to the axis of the beam, at point c in the critical section mn, at a quarter of the depth of the section below the axis of the beam.

Answer: $\sigma_c = 800$ kg/cm^2; $\tau_c = 200$ kg/cm^2; $\sigma_1 = 850$ kg/cm^2; $\sigma_3 = -50$ kg/cm^2; $\sigma_a = 26$ kg/cm^2; $\tau_a = 248$ kg/cm^2.



PROBLEM 4.107

4.108. A simply supported I-beam (I No. 24b) is tested in flexure by a force applied at the mid-span in equal increments $\Delta P = 4$ t.

Extensometers, located on the left from the critical section, on the level of the beam axis, and forming an angle of 45° with it, give a mean increment $\Delta A = 1{\cdot}53$ mm of readings (corresponding to the load increments). The gauge length b of the extensometers is 20 mm, the magnification k is 1200. Assuming the value of the modulus of elasticity $E \approx 2 \times 10^6$ kg/cm^2, find the value of Poisson's ratio μ of the material.

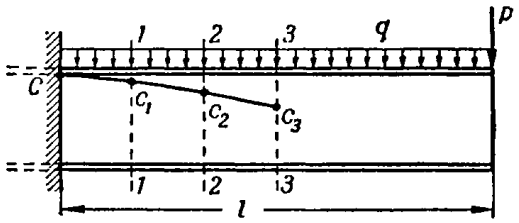
Instruction. The tensile strain of the fibre measured by the extensometer is $\varepsilon = \Delta A/kb$. On the other hand, in terms of the principal stresses $\sigma_1 = -\sigma_3 = \tau$ (from the condition for the material to be in pure shear) the strain is expressed as $\varepsilon = \sigma_1/E - \mu(\sigma_3 E) = \tau/E(1 + \mu)$.

Answer: $\mu = 0{\cdot}3$.

4.109. An I-section (I No. 24a) cantilever loaded by a force $P = 5$ t at the free end carries a continuous load $q = 2$ t/m over the whole length $l = 1$ m. Plot the trajectory of the maximum principal stress passing through the point C of the clamped section where the web joins the flange.

Instruction. Plot the Q and M diagrams. Then calculate σ and τ at the given point C, and find the magnitude and the direction of the principal stress σ_{1c} at this point. Draw four or five intermediate sections along the length of the beam, and find the values of Q

and M for them. Continue the direction of σ_1 found at the point C until it intersects section 1–1 (point C_1). Calculate σ and τ for this point and, having determined the direction of the principal stress σ_{c1}, continue it until it intersects section 2–2, etc. Construct a smooth curve from the broken line obtained.



PROBLEM 4.109

4.110. Using the data of the preceding problem plot the trajectory of the principal compressive stress σ_3 passing through the point where the neutral axis intersects the line 3–3 drawn through the mid-span of the beam.

19. CALCULATION OF COMPOSITE BEAMS

4.111. To reinforce a No. 60a I-beam two horizontal $240 \times 20\,\text{mm}^2$ plates are welded to its flanges. If the allowable shear stress $[\tau_l]$ of the weld is $900\,\text{kg/cm}^2$ and Q is 70 t, determine the required leg width of the welded seam (continuous along the length of the beam). For the same value of Q find also the maximum shear stress in the web of the reinforced I-section.

Answer: $\delta \approx 10$ mm; $\max \tau = 950\,\text{kg/cm}^2$.

4.112. Check the strength of a simply supported welded beam loaded by a continuous uniformly distributed load $q = 4$ t/m, and by two concentrated forces $P = 48$ t applied at equal distances $a = 1$ m from the respective supports. The span l of the beam is 8·0 m. The cross-section is shown in the figure. The dimensions are given in mm. The leg width of the seam is 5 mm. Assume the allowable stress $[\sigma]$ as $1600\,\text{kg/cm}^2$ in tension and in compression, and the allowable stress $[\tau]$ as $1000\,\text{kg/cm}^2$ in shear (both in the base metal and in the weld).

Solution. Having plotted the M and Q diagrams, we find that

$$\max M = 80\,\text{tm}; \quad \max Q = 64\,\text{t}.$$

At the point of application of the force

$$M_1 = 62\,\text{tm}; \quad Q_1 = 60\,\text{t}.$$

The second moment of area, and the section modulus about the neutral axis y, are respectively

$$J_y = \frac{1 \times 80^3}{12} + 2\left[\frac{24 \times 2^3}{12} + (242 \times 41^2)\right] \approx 214{,}000\ \text{cm}^4;$$

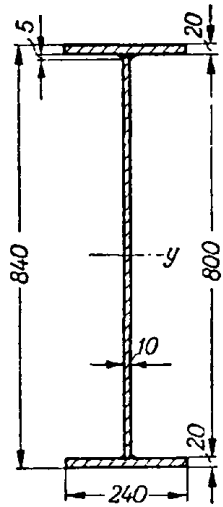
$$W_y = \frac{214{,}000}{42} = 5100\ \text{cm}^3.$$

The first moments of area about the y axis are:

(a) for one of the flanges: $S_{f1} = 2 \times 24 \times 41 = 1970\ \text{cm}^3$;

(b) for one half of the cross-section:

$$S = 1970 + (1 \times 40 \times 20) = 2770\ \text{cm}^3.$$



PROBLEM 4.112

The check on the strength of the cross-section:

(a) for the normal stress:

$$\max \sigma = \frac{\max M}{W} = \frac{80 \times 10^5}{5100} = 1570 \text{ kg/cm}^2 < 1600 \text{ kg/cm}^2.$$

(b) for the shear stress:

$$\max \tau = \frac{QS_{\max}}{J_y \delta} = \frac{64{,}000 \times 2770}{214{,}000 \times 1} = 828 \text{ kg/cm}^2 < 1000 \text{ kg/cm}^2.$$

(c) for the principal stress occurring in the section at the level of the flanking seams, at the point of application of the force:

The stress components are

$$\sigma_c = \frac{M_1 z_c}{J_y} = \frac{62 \times 10^5 \times 40}{214{,}000} = 1160 \text{ kg/cm}^2;$$

$$\tau_c = \frac{Q_1 S_{fl}}{J_y \delta} = \frac{60{,}000 \times 1970}{214{,}000 \times 1} = 552 \text{ kg/cm}^2.$$

The effective stress, according to the shear strain energy theory, is

$$\sigma = \sqrt{\sigma_c^2 + 3\tau_c^2} = \sqrt{1160^2 + (3 \times 552^2)} =$$

$$= 1510 \text{ kg/cm}^2 < 1600 \text{ kg/cm}^2.$$

The maximum shearing force per 1 cm length of the seam is

$$N = \frac{QSe}{J} = \frac{64{,}000 \times 1970 \times 1}{214{,}000} = 590 \text{ kg}.$$

The check on the strength of the welds.

The cross-sectional area of the two seams resisting shearing of the flange

$$F_e = 2 \times 0{\cdot}7\delta \times e = 1{\cdot}4 \times 0{\cdot}5 \times 1 = 0{\cdot}7 \text{ cm}^2.$$

The stress in the weld

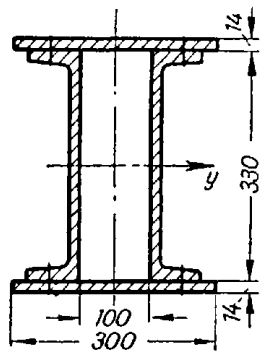
$$\tau_e = \frac{N}{F_e} = \frac{QS_{fl}e}{J \times 2 \times 0{\cdot}7\delta \times e} = \frac{590}{0{\cdot}7} =$$

$$= 840 \text{ kg/cm}^2 < 1000 \text{ kg/cm}^2.$$

Thus the beam has satisfied all strength requirements.

4.113. A simply supported beam, with a span $l = 6$ m, consists of two No. 33 b channels reinforced by two 14×300 mm^2 plates riveted to the flanges (see figure). What forces P can be applied at equal distances $a = 25$ cm from the respective supports, if the maximum normal stresses are not to exceed an allowable stress $[\sigma]$ of 1600 kg/cm^2? What is the maximum acceptable pitch, e, of the rivets, with a diameter $d = 23$ mm, for the limiting strength condition in shear, if $[\tau]$ is 1000 kg/cm^2?

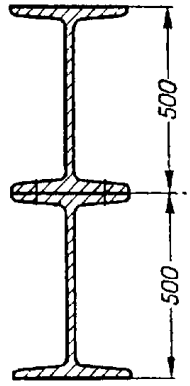
Answer: $P = 136$ t.



PROBLEM 4.113

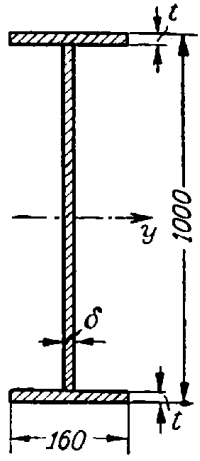
4.114. The main beams of a temporary bridge, with a span $l = 11{\cdot}5$ m, consisting of two No. 50 b I-sections placed on top of each other and riveted together at the flanges, are investigated for bending strength under uniformly distributed load q. The allowable stress $[\sigma]$ is 1650 kg/cm^2. Determine the maximum possible pitch of the rivets for the limiting strength condition in shear, if $[\tau]$ is 900 kg/cm^2. The diameter d of the rivets is 23 mm.

Answer: $e \approx 20$ cm.



PROBLEM 4.114

4.115. Compare the load-carrying capacity of a composite beam of two No. 50b I-sections (see the figure of the preceding problem) with the load-carrying capacity of a welded beam, all dimensions of which (the height, the width, and the thickness of the flanges and the web) are chosen to be the same as the corresponding dimensions of the beam in Problem 4.114. Compare also the weights of the beams.



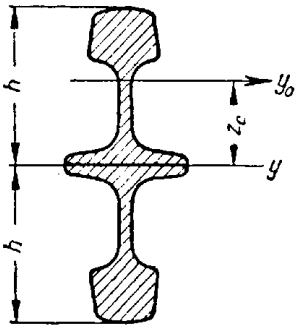
PROBLEM 4.115

Answer: The load-carrying capacity of the beam of two I-sections is 2·5% higher, and the weight 30% greater than the respective quantities of the welded beam.

4.116. Find the magnitude of the shearing force per rivet of a simply supported rail section carrying a continuous uniformly distributed load $q = 1{\cdot}8$ t/m.

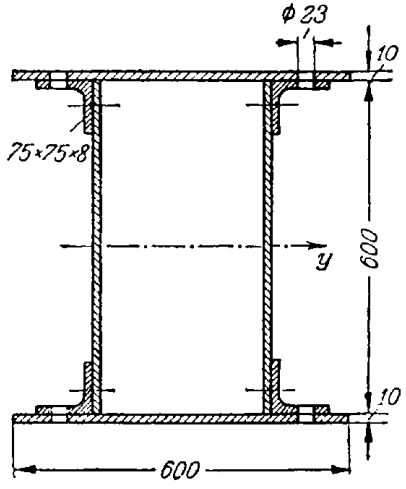
The section consists of two rails of type Ia riveted together at their bases. The cross-sectional area F of one rail is 55·6 cm², the depth h is 140 mm, the distance z from the base to the centroid of the cross-section of the rail is 69·6 mm; the second moment of area J_y of the cross-section of the rail about the centroidal y_0 axis is 1476 cm⁴. The span l of the beam is 6·5 m, the pitch e of the rivets is 150 mm. Determine also the value of the maximum normal stresses.

Answer: $N \approx 2\,t$;
$\sigma_{max} = 1590$ kg/cm².



PROBLEM 4.116

4.117. A composite beam of box section riveted together consists of 10 × 600 mm² plates and 75 mm × 75 mm × 8 mm angles (see



PROBLEM 4.117

figure). It is subjected to bending in the vertical plane. The bending moment M at the critical section is 80 t m, the shear force Q is 48 t. Check the strength of the cross-section, if the allowable stresses are: $[\sigma] = 1600$ kg/cm^2 and $[\tau] = 1000$ kg/cm^2. Assume that due to the rivet holes the cross-section is weakened by 15%.

Answer: $\max \sigma = 1590$ kg/cm$^2 \approx 1600$ kg/cm^2;
$\max \tau = 440$ kg/cm$^2 < 1000$ kg/cm^2.

4.118. Check the strength in shear, and in compression, of the horizontal outside rivets of the cross-section represented in the figure of Problem 4.117. The diameter d of the rivets is 23 mm, the allowable stress $[\tau]$ is 1000 kg/cm^2, and $[\sigma_c]$ is 2800 kg/cm^2. The pitch a of the rivets is 120 mm, the shear force Q is 48 t, and the gross second moment of area J_y of the cross-section is 183,900 cm^4. Compare also the value of the shearing force per rivet in the horizontal rivets with that in the vertical rivets.

Answer: $\tau = 930$ kg/cm$^2 < 1000$ kg/cm^2;
$\sigma_c = 2100$ kg/cm$^2 < 2800$ kg/cm^2.

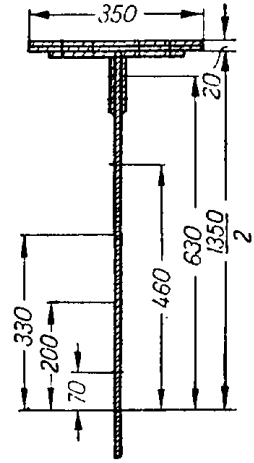
The force taken by the horizontal rivets is 35% greater than that taken by the vertical rivets.

4.119. Check the strength of a simply supported riveted I-beam, with the span $l = 12$ m, loaded by five concentrated forces $P = 25$ t applied symmetrically at every 2 m along the length of the beam. The weight q of the beam is taken as 0·4 t/m. The cross-section of the beam is shown in the figure (the dimensions are given in mm). The pitch e of the rivets is 130 mm, the diameter d is 20 mm. Take the allowable stresses as follows:

$$[\sigma] = 1400 \text{ kg/cm}^2;$$
$$[\tau] = 800 \text{ kg/cm}^2;$$
$$[\sigma_c] = 2400 \text{ kg/cm}^2.$$

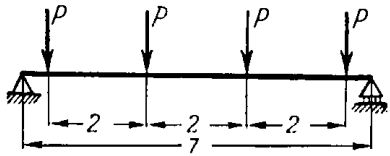
Instruction. Take into consideration the fact that each angle is weakened by two rivet holes: one vertical and the other horizontal, and that each horizontal plate is weakened by two vertical rivet holes.

Answer: $\max M = 232{\cdot}2$ tm; $J_{\text{gross}} = 1{,}295{,}000$ cm^4;
$\max Q = 64{\cdot}9$ t; $J_{\text{net}} = 1{,}128{,}100$ cm^4;
$S_{\text{fl}} = 8100$ cm^3; $\max S = 10{,}400$ cm^3;
$\max \sigma = 1435$ kg/cm^2; $\max \tau = 600$ kg/cm^2.



PROBLEM 4.119

4.120. Select a cross-section for a simply supported welded I-beam (see diagram) loaded by four symmetrically placed forces $P = 24$ t. The span l of the beam is 7 m. The dimensions in the figure are given in metres. The allowable bending stress $[\sigma]$ is 1400 kg/cm^2 in tension and in compression; the stress $[\tau]$ is 1000 kg/cm^2 in shear of the parent metal; the stress $[\tau_e]$ is 800 kg/cm^2 in shear of the welds.



PROBLEM 4.120

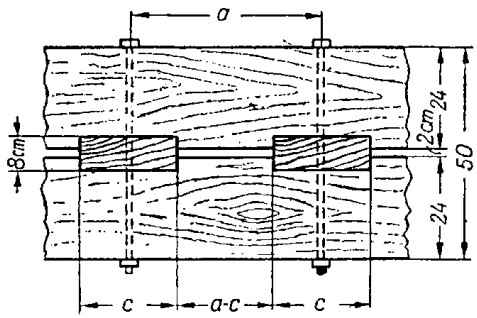
Instruction. For a detailed instruction see Chapter 16 of the book *Strength of Materials* by N. M. Belyayev. The depth of the web may be determined within the limits of $l/8$ and $l/12$ by the empirical formula $h \approx 1{\cdot}3\sqrt{W}$ to $h \approx 1{\cdot}5\sqrt{W}$, (where W is in cm^3).

The wall thickness, by the stipulation of stability, must not be less than $\delta = \sqrt{h}/12{\cdot}5$ cm. The width of the flanges is taken as $b = 0{\cdot}25h$ to $b = 0{\cdot}4h$, but not more than thirty times the thickness. The thickness itself is chosen so as to obtain the required section modulus.

Having fixed the dimensions of the cross-section we shall carry out the check on its strength (see Problem 4.112). If the deviations from the allowable stresses exceed $\pm 5\%$, the dimensions of the crosssection must be accordingly increased or reduced. This is done most simply by altering the width of the flanges. The leg width of the weld is determined from the calculation for shear $\delta_e \geq QS_{f1}/1{\cdot}4\,J[\tau]$.

4.121. A wooden composite beam consists of two pine logs 20×24 cm^2 in cross-section, joined to each other by means of bolts and pine blocks of rectangular cross-section (see figure). The beam is simply supported and loaded by a concentrated force $P = 6$ t applied at the mid-span. The span l of the beam is 6·0 m. Determine the required dimensions and the number of blocks, if the allowable stress $[\sigma_c]$ in compression along the grain of the pine ist 70 kg/cm^2, and the stress $[\tau]$ in shear is 12 kg/cm^2. It is assumed that the bolts do not resist shear.

Answer: $a \approx 45$ cm; $c = 17{\cdot}5$ cm; $n \approx 14$.



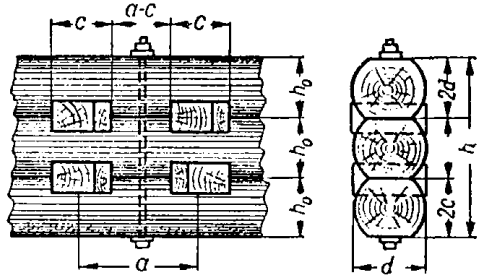
PROBLEM 4.121

4.122. Determine the required dimensions and the number of oak keying blocks for the main beam of a temporary bridge having a span $l = 6$ m. The beam consists of three logs cut away at

both sides and joined to each other by means of keying blocks and bolts, with a diameter d of 16 mm (see figure).

The diameter of the logs is 22 cm, the thickness of the slab cut away is 2·5 cm. The cross-sectional area F of one log is 390 cm², the second moment of area J_y is 8800 cm⁴. The load q is 1·1 t/m. Assume that the depth d_k of the cuts for the keying block is 3 cm.

Check also the strength of the beam, taking into account the weakening effect due to the bolt holes.



PROBLEM 4.122

Allowable stresses:

(a) for the material of the beam:
the flexural stress $[\sigma]$ is 80 kg/cm² * in tension and in compression; the bearing stress $[\sigma_c]$ is 70 kg/cm² along the fibre; the shear stress $[\tau]$ is 12 kg/cm² along the fibre;

(b) for the material of the keying block:
the bearing stress $[\sigma_{90}]$ is 50 kg/cm² across the grain; the shear stress $[\tau_{90}]$ is 10 kg/cm² across the grain.

Answer: $n = 18$; $c = 15$ cm; $\sigma = 54$ kg/cm².

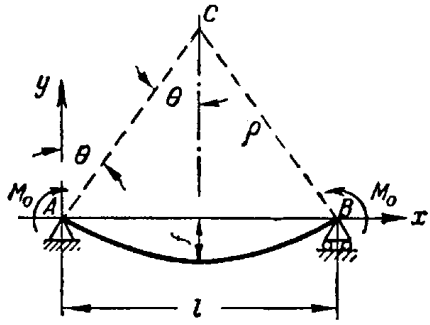
* The basic allowable stress is reduced, since a composite beam is less efficient than a single one, due to the unavoidability of bars being displaced along each other.

CHAPTER 5

DETERMINATION OF DEFORMATIONS ACCOMPANYING FLEXURE, AND CALCULATION OF STATICALLY INDETERMINATE SYSTEMS*

20. ANALYTICAL METHOD OF DETERMINING DEFORMATIONS

5.1. A steel straight-edge with a span l of 5 m, having a $5 \times 60\,\text{mm}^2$ cross-section, is bent by two couples $M_0 = 100$ kg cm applied to its ends (see figure). Establish the equation to the curve along which the straight-edge is bent. Find the maximum slope and deflection of the straight-edge.



PROBLEM 5.1

Solution. We shall find the values of the angular and linear displacements (θ and f) by integrating the approximate differential equation of flexure

$$EJy'' = M(x) = M_0. \tag{1}$$

* See also Appendix B, Section 3.

Integrating twice, we have

$$EJy' = M_0 x + C;$$

$$EJy = \frac{M_0 x^2}{2} + Cx + D.$$

The constants C and D will be found from the end conditions:

$$\text{at } x = 0 \quad y = 0 \quad \text{and} \quad D = 0;$$

$$\text{at } x = l \quad y = 0 \quad \text{and} \quad C = -\frac{M_0 l}{2}.$$

Substituting the values of these constants we have

$$y' = \frac{M_0}{EJ}\left(x - \frac{l}{2}\right); \tag{2}$$

$$y = \frac{M_0}{2EJ}(x^2 - lx) \tag{3}$$

(equation of a parabola).

In these equations the second moment of area of the cross-section

$$J = \frac{bh^3}{12} = \frac{6 \times 0{\cdot}5^3}{12} = 0{\cdot}0625 \text{ cm}^4.$$

The maximum slope will occur at one of the supports; for example, at $x = 0$ the equation (2) gives

$$\theta_A = -\frac{M_0 l}{2EJ} = -\frac{100 \times 100}{2 \times 2 \times 10^6 \times 0{\cdot}0625} = -0{\cdot}04;$$

(the minus sign indicates that the section at support A rotates clockwise).

The maximum deflection will occur at the mid-span, i.e. $\max y = f$ at $x = \frac{l}{2}$:

$$f = -\frac{M_0 l^2}{8EJ} = -\frac{100 \times 100^2}{8 \times 2 \times 10^6 \times 0{\cdot}0625} = -1 \text{ cm};$$

(here the minus sign indicates the fact that the direction of the deflection does not coincide with the positive direction of the y axis).

The results obtained are not completely accurate since they have been found from the approximate equation (1). An accurate solution may be obtained by the following considerations.

Since $M(x) = M_0 = \text{const}$, the curvature of the elastic curve $1/\varrho = M(x)/EJ$ will be constant.

Consequently, the straight-edge deflects not along a parabola but along an arc of a circle of the radius

$$\varrho = \frac{EJ}{M_0} = \frac{2 \times 10^6 \times 0{\cdot}0625}{100} = 1250 \text{ cm}.$$

The slope of the end section will be determined as the ratio of the arc to the radius

$$\theta = \frac{1}{2\varrho} = \frac{100}{2 \times 1250} = 0{\cdot}04.$$

The deflection at the mid-span will be determined as the depth of an arc of length l (measured from the chord)

$$f = \varrho(1 - \cos\theta) = \varrho\left(1 - \cos\frac{1}{2\varrho}\right) =$$
$$= 1250(1 - 0{\cdot}999206) = 0{\cdot}993 \text{ cm}.$$

The deviation between the approximate and accurate solutions is about 0·7%.

5.2. A rod with a length l of 1 m and a cross-section of 2×2 cm^2, clamped at one end, is bent by a couple M_0 of 10 kgm applied to the other end. Find the value of the modulus of elasticity of the material, and the radius of curvature of the beam, if the slope θ of the end section is 0·0375.

Answer: $E = 2 \times 10^6$ kg/cm^2; $\varrho = 26{\cdot}75$ m.

5.3. A simply supported beam with a span of 2 m is bent along an arc of a circle. Its deflection at the mid-span is 0·5 cm. If E is

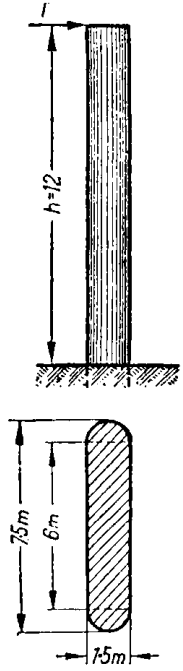
1×10^5 kg/cm^2 and J is 230 cm^4, determine the radius of curvature of the beam, and the value of the bending moment.

Answer: $\varrho = 100$ m; $M = 2300$ kg cm.

5.4. To determine the modulus of elasticity of the brickwork of a bridge pier, the horizontal displacement of its upper section, due to the action of a braking force $T = 24$ t, was measured. The deflection f was found to be 8·75 mm.

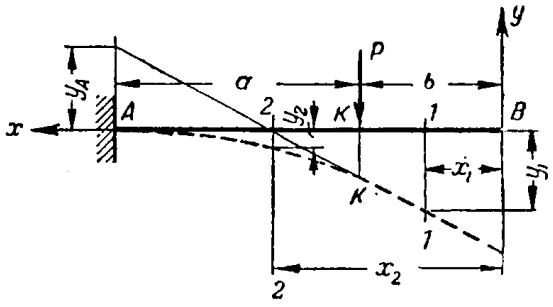
What is the modulus of elasticity of the material, if the height h of the pier above the foundation is 12 m, and the dimensions of the cross-section are as shown in the figure? Neglect the deformations of the foundation and the ground.

Answer: $E = 8{\cdot}08 \times 10^4$ kg/cm^2.



PROBLEM 5.4

5.5. Find the slope and deflection of the end section of a cantilever with a span l of 3 m, loaded by a concentrated load P of 1 t applied at a distance $a = 2$ m from the constraint (see figure). The cross-section of the cantilever is I No. 16 b. The material is Steel 3.



PROBLEM 5.5

Solution. We shall set up the differential equations of flexure for both sections, using the same origin (the right-hand end of the beam), and carry out the integration without opening the brackets:

First section

$$EJy_1'' = 0; \tag{1}$$

$$EJy_2' = C_1; \tag{a}$$

$$EJy_1 = C_1x_1 + D_1. \tag{b}$$

Second section

$$EJy_2'' = -P(x_2 - b); \tag{2}$$

$$EJy_2' = -\frac{P}{2}(x_2 - b)^2 + C_2; \tag{c}$$

$$EJy_2 = -\frac{P}{6}(x_2 - b)^3 + C_2x_2 + D_2. \tag{d}$$

From the condition of compatibility, i.e. for the adjacent sections of the beam to meet at point K, which is common to them both, it follows that at $x_1 = x_2 = b$

$$\theta_K = y_1' = y_2' \qquad \text{and} \qquad f_K = y_1 = y_2.$$

Hence from the equation (a) and (c) we shall find that $C_1 = C_2 = C$; from the equations (b) and (d) we shall find that $D_1 = D_2 = D$.

The values of the arbitrary constants C and D are determined from the end conditions of the beam. At the fixed end both the

angular and linear displacements are zero; i.e. at $x_2 = l$, $y_2' = 0$ and $y_2 = 0$, whence

$$C = \frac{P(l-b)^2}{2} = \frac{Pa^2}{2}; \quad D = \frac{P(l-b)^3}{6} - \frac{Pa^2 l}{2} = \frac{Pa^2}{6}(a - 3l).$$

Now the displacements of the end section of the beam can be determined from the equations (a) and (b) of the first section, with $x_1 = 0$:

$$\theta_B = y_1' \quad \text{at} \quad x_1 = 0;$$

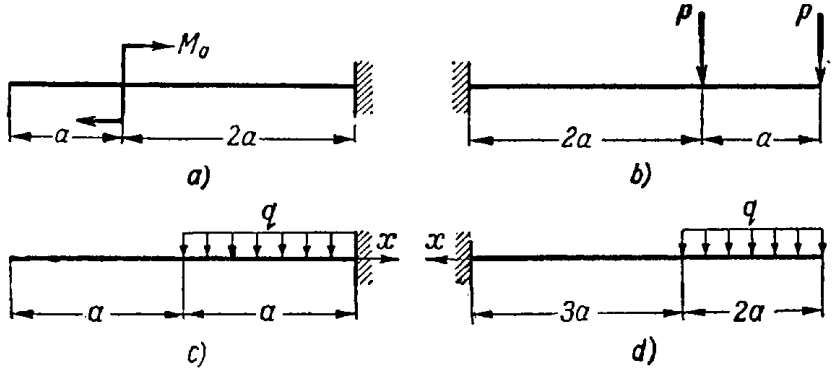
$$\theta_B = \frac{P_a^2}{2EJ} = \frac{1000 \times 200^2}{2 \times 2 \times 10^6 \times 1130} = 0{\cdot}00885;^*$$

$$f_B = y_1 \quad \text{at} \quad x_1 = 0;$$

$$f_B = \frac{Pa^2(a - 3l)}{6EJ}$$

$$= \frac{1000 \times 200^2}{6 \times 2 \times 10^6 \times 1130}(200 - (3 \times 300)) = -2{\cdot}06 \text{ cm}.$$

5.6. Integrating the differential equation of flexure, find the angular and linear displacements of the free end of a cantilever loaded as indicated in the figure (see diagrams (a), (b), (c) and (d)).



PROBLEM 5.6

* The plus sign indicates that the rotation of the end section through angle θ_B takes place clockwise, since the x axis is directed from right to left.

Answer:

(a) $\theta = -\frac{2M_0 a}{EJ}$; $\quad f = \frac{4M_0 a^2}{EJ}$;

(b) $\theta = -\frac{13Pa^2}{2EJ}$; $\quad f = -\frac{41Pa^3}{3EJ}$;

(c) $\theta = \frac{qa^3}{6EJ}$; $\quad f = -\frac{7qa^4}{24EJ}$;

(d) $\theta = -\frac{49qa^3}{3EJ}$; $\quad f = -\frac{59qa^4}{EJ}$.

5.7. What camber* must be given to a simply supported steel No. 60a I-beam, if it is to become straight when loaded by a force P of 14 t applied at the middle of the span $l = 12$ m? Neglect the weight of the beam.

Answer: The depth f of the camber is 3·0 cm.

5.8. Select an I-section for a cantilever, which is uniformly loaded over the whole length $l = 2$ m by a load $q = 1$ t/m, with the condition that the deflection of the free end must not exceed the allowable value $[f] = l/450$. Will the beam then satisfy the limiting stress condition, if $[\sigma] = 1400$ kg/cm²?

Answer: I No. 20a; $\max \sigma = 845$ kg/cm².

5.9. A rolled steel No. 45a I-beam is simply supported and has a span of 10 m. The maximum deflection must not exceed 1/600 of the span. What is the maximum intensity of a uniformly distributed load which may be carried by the beam, and what are the corresponding maximum normal stresses?

Answer: $q = 0{\cdot}825$ t/m; $\sigma_{max} = 720$ kg/cm².

5.10. Two beams of uniform cross-section are perfectly similar; all dimensions of one of them, including the span, are twice the corresponding dimensions of the other beam.

Compare the stresses and deflections due to the self-weight. Compare also the load per metre, which may be applied to pro-

* An upward deflection of a beam, carried out beforehand to eliminate sagging under a load, is called the camber.

duce (a) equal maximum normal stresses, and (b) equal ratios of the maximum deflection to the span.

Answer: $\frac{\sigma_1}{\sigma_2} = 1$; $\frac{f_1}{f_2} = \frac{1}{8}$; (a) $\frac{q_1}{q_2} = \frac{1}{2}$; (b) $\frac{q_1}{q_2} = \frac{1}{2}$.

5.11. An I-beam with a depth $h = 60$ cm, one end built into a wall, is bent in the vertical plane by a force P applied to the free end of the beam. Determine the deflection of the free end of the beam, if the force causes a maximum normal stress σ of 1600 kg/cm^2 in the critical section of the beam. The length l of the beam is 2·5 m; the modulus E is 2·1 $\times$ 10^6 kg/cm^2.

Instruction. Express the deflection in terms of the stress and the depth of the beam.

Answer: $f = 0{\cdot}53$ cm.

5.12. A simply supported steel No. 33a I-beam is bent by a force P applied to the mid-span. In the critical cross-section it produces a maximum normal stress σ of 1000 kg/cm^2, and in the neutral layer a shear stress τ of 270 kg/cm^2. Find the span l of the beam, and the value of the deflection f at the section underneath the force.

Answer: $l = 1{\cdot}99$ m; $f = 1{\cdot}0$ mm.

5.13. A steel beam of constant rectangular cross-section, simply supported, is loaded by a uniformly distributed load. The maximum normal stress in the cross-section is 600 kg/cm^2, and the maximum deflection is 1/600 of the span. Determine the ratio of the beam's depth to its span.

Answer: $\frac{h}{l} = 0{\cdot}0375$.

5.14. A steel beam with the depth h and the span l, simply supported, is loaded by a uniformly distributed load q. How many times must the span l be greater than the depth h, if the maximum deflection is not to exceed 1/400 of the span, and the maximum normal stress is not to exceed 1400 kg/cm^2?

Answer: $l = \frac{120}{7}\, h$.

5.15. A wooden cantilever of rectangular cross-section of $20 \times 24\ \text{cm}^2$ is loaded by a continuous load $q(x)$ in the shape of a triangle (see figure (a)), with the resultant $Q = \frac{q_0 l}{2} = 2\,\text{t}$.

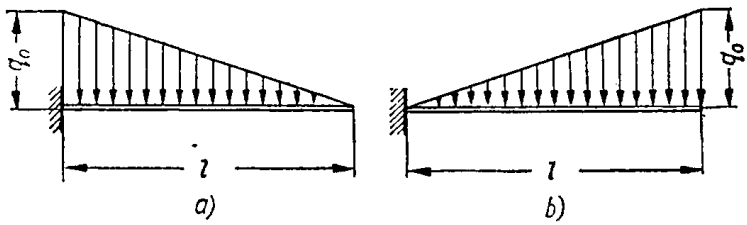
(a) Find the maximum deflection, and the maximum slope, if the length l of the beam is 1·5 m.

(b) By how many times will the value of the maximum normal stress (and that of the maximum deflection) increase, if the beam is loaded by the same triangular load, positioned as indicated in figure (b)?

Answer: (a) $f = -\frac{Ql^3}{15EJ} = 0{\cdot}29\ \text{cm}$;

$$\theta_{\max} = -\frac{Ql^2}{12EJ} = 0{\cdot}0245;$$

(b) $\max \sigma_b = 2 \max \sigma_a$; $f_b = 2{\cdot}75\, f_a$.



PROBLEM 5.15

5.16. A simply supported beam of given span is loaded by a continuous uniformly distributed load. The cross-section of the beam is symmetrical about both principal axes. Find how the maximum deflection of the beam is dependant on the cross-section, if $\sigma_{\max} = [\sigma]$.

Answer: $\max f = \frac{5}{24} \frac{l^2}{E} \frac{[\sigma]}{h}$.

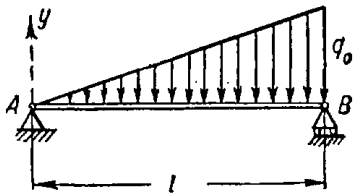
5.17. A steel I-beam with a span $l = 6\,\text{m}$ is loaded by a continuous load $q(x)$ in the shape of a triangle (see figure) having a maximum intensity, q, of 6 t/m. Choose a cross-section for the beam if $[\sigma]$ is $1600\,\text{kg/cm}^2$, and determine the value of the maximum deflection $f_{\max}$ and the slopes θ_A and θ_B at the supports.

Answer: I No. 36a.

$$\theta_A = -\frac{7q_0l^3}{360EJ} = -0{\cdot}008;$$

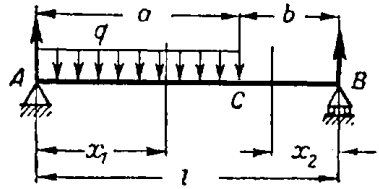
$$\theta_B = \frac{q_0l^3}{45EJ} = 0{\cdot}00915;$$

$$f_{max} = 0{\cdot}0065\,\frac{q_0l^4}{EJ} = 1{\cdot}6\text{ cm} \quad (\text{at } x = 0{\cdot}52l).$$



PROBLEM 5.17

5.18. A timber beam, with a rectangular cross-section of 20×15 cm^2, is simply supported and has a span l of 6 m. A uniformly distributed load Q of 1·2 t is placed on the section of length $a = 4$ m (see figure), adjacent to the left-hand end of the beam. Determine the deflection at mid-span.



PROBLEM 5.18

Solution. The intensity of the continuous load

$$q = \frac{Q}{a} = \frac{1{\cdot}2}{4} = 0{\cdot}3\text{ t/m}.$$

Since the arrangement of the load divides the beam into two parts, we form two differential equations, thereby denoting deflections in the left-hand part by y_1, and those in the right-hand part by y_2. The abscissa for each part is measured from the sup-

ports to meet each other; they are denoted correspondingly by x_1 and x_2. We determine the reactions A and B:

$$A = \frac{qa\left(l - \frac{a}{2}\right)}{l} = \frac{0{\cdot}3 \times 4 \times 4}{6} = 0{\cdot}8 \text{ t};$$

$$B = \frac{qa^2}{2l} = \frac{0{\cdot}3 \times 4^2}{2 \times 6} = 0{\cdot}4 \text{ t}.$$

We set up the differential equations, and integrate them:

$$EJy_1'' = Ax_1 - \frac{qx_1^2}{2}; \qquad EJy_2'' = Bx_2;$$

$$EJy_1' = \frac{Ax_1^2}{2} - \frac{qx_1^3}{6} + C_1; \qquad EJy_2' = \frac{Bx_2^2}{2} + C_2;$$

$$EJy_1 = \frac{Ax_1^3}{6} - \frac{qx_1^4}{24} + C_1x + D_1; \quad EJy_2 = \frac{Bx_2^3}{6} + C_2x + D_2.$$

For the determination of the four constants of integration we have the following conditions: (a) the deflections at the supports are zero; (b) at point C which separates the parts, the deflections are equal in magnitude for both parts, whereas the slopes are equal in magnitude, but have opposite signs. (This is due to the fact that the axes x_1 and x_2 have opposite directions.)

Analytically, these conditions are reduced to the four equations:

(a) at $x_1 = 0$: $y_1 = 0$;

(b) at $x_2 = 0$: $y_2 = 0$;

(c) at $x_1 = a$ and $x_2 = l - a = b$: $\theta_c = -y_1' = y_2'$;

(d) at $x_1 = a$ and $x_2 = l - a = b$: $f_c = y_1 = y_2$.

The first two equations give $D_1 = D_2 = 0$; the third equation is of the form

$$\frac{Aa^2}{2} - \frac{qa^3}{6} + C_1 = -\frac{Bb^2}{2} - C_2;$$

and the fourth equation is reduced to

$$\frac{Aa^3}{6} - \frac{qa^4}{24} + C_1 a = \frac{Bb^3}{6} + C_2 b.$$

Since in the solution of the problem the method of equating arbitrary constants of integration was not employed, it will be necessary to solve the two simultaneous equations for the determination of these constants. The result is as follows:

$$C_1 = -\frac{1}{a+b}\left[\frac{Aa^2}{6}(a+3b) + \frac{Bb^3}{3} - \frac{qa^3}{24}(a+4b)\right]$$

$$= -\frac{1}{6}\left[\frac{0{\cdot}8 \times 4^2}{6}\{4 + (3 \times 2)\} + \frac{0{\cdot}4 \times 2^3}{24}\{4 + (4 \times 2)\}\right]$$

$$= -2{\cdot}133 \text{ tm}^2;$$

$$C_2 = -\frac{1}{a+b}\left[\frac{Aa^3}{3} + \frac{Bb^2}{6}(3a+b) - \frac{qa^4}{8}\right]$$

$$= -\frac{1}{6}\left[\frac{0{\cdot}8 \times 4^3}{3} + \frac{0{\cdot}4 \times 2^2}{6}\{(3 \times 4) + 2\} - \frac{0{\cdot}3 \times 4^4}{9}\right]$$

$$= -1{\cdot}867 \text{ tm}^2.$$

For the calculation of the deflection f at the mid-span it is necessary to insert $x_1 = 2/l = 3$ m in the equation; then

$$EJf = 0{\cdot}8\frac{3^3 \times 0{\cdot}3 \times 3^4}{6 \times 24} - (2{\cdot}133 \times 3)$$

$$= -3{\cdot}812 \text{ tm}^3 = 3812 \times 10^6 \text{ kgcm}^3.$$

The second moment of area of the cross-section and the flexural rigidity are respectively

$$J = \frac{20^3 \times 15}{12} = 10^4 \text{ cm}^4$$

and $$EJ = 1 \times 10^5 \times 10^4 = 10^9 \text{ kgcm}^2.$$

Consequently,

$$f = \frac{3812 \times 10^6}{10^9} = 3{\cdot}81 \text{ cm}.$$

We shall also find the position of the maximum deflection. At $y = y_{\max}$ $\frac{dy}{dx} = 0$, i.e. $EJy_1' = \frac{Ax_1^2}{2} - \frac{qx_1^3}{6} + C_1 = 0$, whence $x_1 = 2{\cdot}889$ m and $f = \max y_1 = 3{\cdot}818$ cm.

Thus, the deflection at the middle of the span differs only slightly from the maximum value.

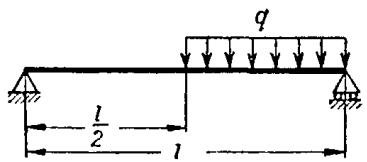
The ratio of the deflection to the span is

$$\frac{f}{l} = \frac{3{\cdot}82}{600} = \frac{1}{158} = 0{\cdot}00635.$$

In order to simplify the problem of determining the arbitrary constants we advise the student to solve the same example making use of the method of equating arbitrary constants of integration.

5.19. A timber beam, having a rectangular cross-section of 18×20 cm^2, and a span l of 4 m, is simply supported and loaded by a continuous load $Q = 2$ t, uniformly distributed over one half of the beam. Find the deflection at the middle of the span, and the slopes at the supports.

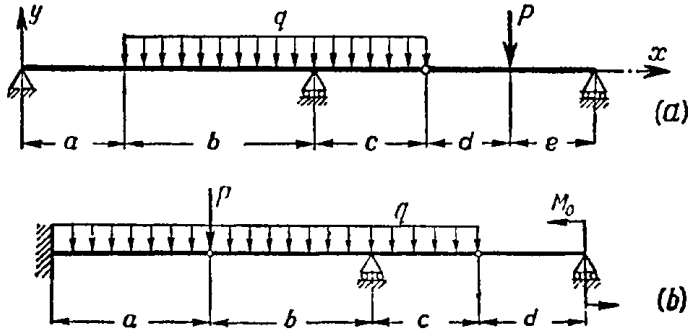
Answer: $f_{\max} = \frac{5Ql^3}{384EJ} = -1{\cdot}39$ cm.



PROBLEM 5.19

5.20. How many differential equations must be integrated in order to determine the deflection of the point of application of the load

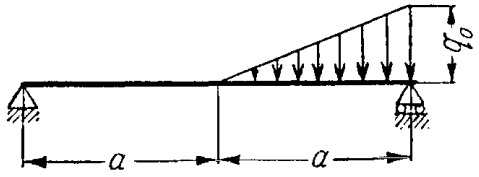
of the beams represented in figures (a) and (b)? How many arbitrary constants of integration must be determined? Write down the required conditions for their determination. How will the answers be affected, if in the problem it is required to find the slope at the right-hand end of the beam?



PROBLEM 5.20

5.21. Find the deflection at the mid-span of the beam shown in the figure. $l = 2a = 6$ m; $q_0 = 4$ t/m. Select the cross-section (an I-profile) on the maximum bending moment, with $[\sigma] = 1600$ kg/cm^2. The material is Steel 3.

Answer: $f = -\dfrac{3q_0a^4}{80EJ}$.

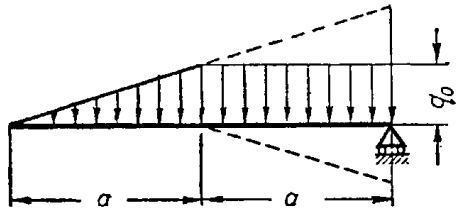


PROBLEM 5.21

5.22. Find the deflection at the mid-span of the beam shown in the figure. Make use of the method of equating arbitrary constants of integration of the differential equations of flexure.

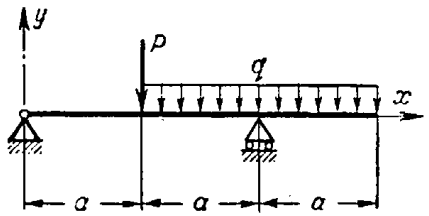
Instruction. Load the beam by the additional triangular loads, directed upwards and downwards as shown by the broken lines in the figure.

Answer: $f = -\frac{4{\cdot}1 q_0 a^4}{24EJ} \approx -\frac{q_0 a^4}{6EJ}$.



PROBLEM 5.22

5.23. Making use of the method of initial parameters, find the deflection of the free end of the beam represented in the figure.



PROBLEM 5.23

Solution. The general equation of flexure, by the method of initial parameters, has the form

$$EJy = EJf_0 + EJ\theta_0 + \frac{M_0 x^2}{2!} + \frac{Q_0 x^3}{3!} + \sum \frac{M_0 (x - a_i)^2}{2!} +$$

$$+ \sum \frac{P_i (x - b_i)^3}{3!} + \sum \frac{q_i (x \quad c_i)^4}{4!},$$

where f_0, θ_0, Q_0, and M_0 are the initial parameters, i.e. the geometrical and force factors at the origin (at $x = 0$), which may be determined from the end conditions of the beam; M_i, P_i, q_i are the given force factors (including also the reactions at the supports); a_i and b_i are the distances from the origin to the section at which the concentrated forces and the couples are applied, and c_i is the distance to the section where the continuous load begins.

In the case being considered

$$A = \frac{P}{2}; \qquad B = \frac{P}{2} + 2qa.$$

The initial parameters at the origin (at point A) correspondingly will be:

$$M_0 = 0; \qquad Q_0 = A = \frac{P}{A}; \qquad f_0 = 0;$$

θ_0 must be determined from the condition that at $x = 2a$, $y = 0$. The equation of the elastic curve will have the form

$$EJy = EJ\theta_0 + \frac{P}{2}\frac{x^3}{3!} - \frac{P(x - a_1)}{3!} + \left[\left(\frac{P}{2} + 2qa\right)\frac{(x - 2a)^3}{3!}\right] - \frac{q(x - a)^4}{4}.$$

Putting $x = 2a$, we find that

$$0 = (EJ\theta_0 \times 2a) + \frac{P}{2}\frac{(2a)^3}{1 \times 2 \times 3} - \frac{Pa^3}{1 \times 2 \times 3} - \frac{qa^4}{1 \times 2 \times 3 \times 4},$$

whence

$$EJ\theta_0 = \frac{qa^3}{48} - \frac{Pa^2}{4}.$$

Now substituting this value for the initial parameter θ_0, and putting $x = 3a$, we shall find the required displacement of the free end D of the beam:

$$EJf_D = \left[\left(\frac{qa^3}{48} - \frac{Pa^2}{4}\right) \times 3a\right] + \frac{P}{2}\frac{(3a)^3}{6} - \frac{P}{6}(2a)^3 + \left(\frac{P}{2} + 2qa\right)\frac{a}{6} - q\frac{(2a)^4}{24}.$$

Hence

$$f_D = \frac{Pa^3}{4EJ} - \frac{13qa^4}{48EJ}.$$

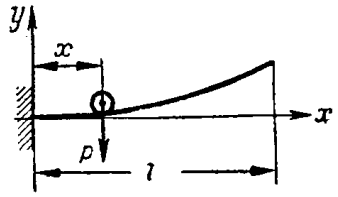
5.24. Using the method of initial parameters, establish the curve to which a simply supported beam of span l should be bent before-

hand, so that, after being loaded by a uniformly distributed load q, it would become straight.

Answer: $y = -\frac{ql^4}{24EJ}\left(\frac{x^4}{l^4} - \frac{2x^3}{l^3} + \frac{x}{l}\right)$.

5.25. A load P moves along a cantilever (see figure). To what curve must the beam be preset, if the load is to remain at the same height during the movement?

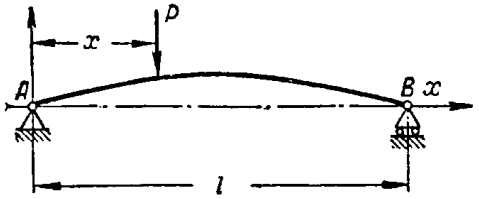
Answer: $y = \frac{Px^3}{3EJ}$.



PROBLEM 5.25

5.26. To what curve must a simply supported beam AB be preset, so that during the displacement of a load P along the span its point of application would remain at the level of the supports A and B (see figure)?

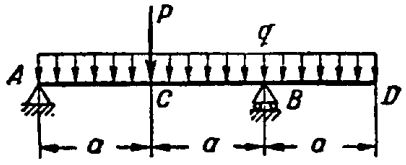
Answer: $y = \frac{Px^2}{3EJl}(x^2 - 2lx + l^2)$.



PROBLEM 5.26

5.27. By the method of initial parameters find the deflection at the mid-span and at the overhung end of the beam represented in the figure. $P = 2$ t; $q = 1$ t/m; $E = 2{\cdot}1 \times 10^6$ kg/cm^2; $a = 2$ m. The cross-section is I No. 18.

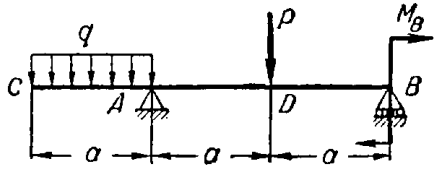
Answer: $f_C = -1{\cdot}20$ cm; $f_D = 0{\cdot}6$ cm.



PROBLEM 5.27

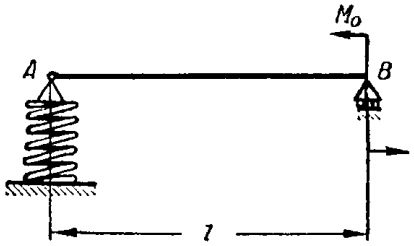
5.28. Using the method of initial parameters find the deflection of the overhung end of the timber beam shown in the figure. $q = 400$ kg/m, $a = 2$ m, $P = qa = 800$ kg, $M_B = Pa$; the dimensions of the cross-section are 16×20 cm^2.

Answer: $f = -3{\cdot}25$ cm.



PROBLEM 5.28

5.29. The beam AB, with a span l, is supported with its right-hand end on a hinged support, and its left-hand end on a helical spring. It is bent by a couple applied at point B (see figure). Set up the equation of flexure for the beam, if the radius of the spring is R, the radius of the rod of the spring is r, and the number of coils is n.



PROBLEM 5.29

Instruction. At $x = 0$ the deflection is equal to the compression of the spring ($y_A = -\lambda$).

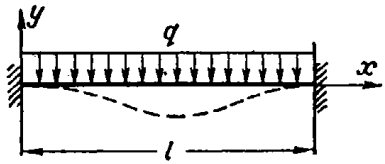
$$\textit{Answer: } y = \frac{M_0 x}{6EJl}(x^2 - l^2) + \frac{\lambda}{l}x - \lambda, \quad \text{where}$$

$$\lambda = \frac{4M_0R^3n}{Glr^4}.$$

5.30. Using the method of initial parameters find the deflection at the mid-span for a beam clamped at both ends and loaded by a continuous uniformly distributed load (see figure). Assume that the longitudinal reactions are zero.

Instruction. Choosing the co-ordinate axes as shown in the figure, we note that the initial deflection and slope are zero. From the condition of symmetry $Q_0 = A = \frac{ql}{2}$, and M_0 is to be found from the condition that at $x = l$, $y = 0$.

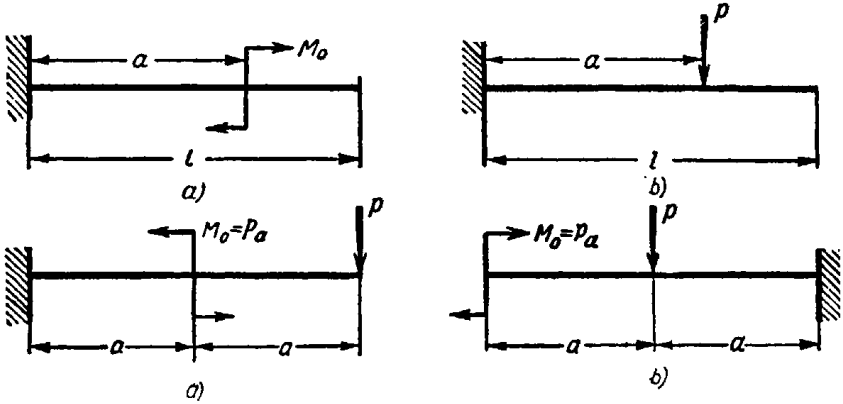
Answer: $f = -\frac{ql^4}{384EJ}$.



PROBLEM 5.30

21. GRAPHICAL AND SEMI-GRAPHICAL METHODS OF DETERMINING DEFORMATIONS

5.31. Using the semi-graphical method to determine displacements, find the values of the deflections and the slopes of the end sections of the cantilevers shown in the figure.



PROBLEM 5.31

Answer: (a) $f = \dfrac{M_0 a}{2EJ}(a - 2l)$, $\theta = -\dfrac{M_0 a}{EJ}$;

(b) $f = \dfrac{Pa^2}{6EJ}(a - 3l)$, $\theta = -\dfrac{Pa^2}{2EJ}$;

(c) $f = -\dfrac{7}{6}\dfrac{Pa^3}{EJ}$, $\theta = -\dfrac{Pa^2}{EJ}$;

(d) $f = -\dfrac{Pa^3}{3EJ}$, $\theta = \dfrac{3}{2}\dfrac{Pa^2}{EJ}$.

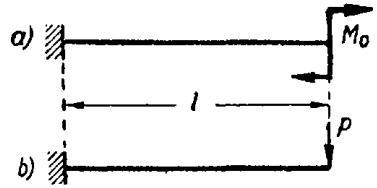
5.32. A steel No. 40a I-section cantilever is bent by a force P applied to the free end. What must be the value of the force P, and the length l of the beam for the deflection f of the end section to be 1 cm, and the slope θ to be 0·005?

Answer: $l = 3{\cdot}0$ m; $P = 4{\cdot}83$ t.

5.33. (1) In which of the following two cases of loading are the normal stresses induced in a cantilever of given length and cross-section greater? The end deflections are the same in each case (see figure): (a) when loaded by a couple M_0, or (b) when loaded by a concentrated force P?

(2) Compare the deflection of the free end of the beam in case (a) with that in case (b), if the maximum normal stress is equal to the allowable stress.

Answer: (1) In case (b), by 50%; (2) $f_a = \dfrac{3}{2} f_b$.



PROBLEM 5.33

5.34. Determine by the semi-graphical method the deflection at points D and C of beam ABC shown in the figure, and also the slopes θ_A and θ_B at the supports.

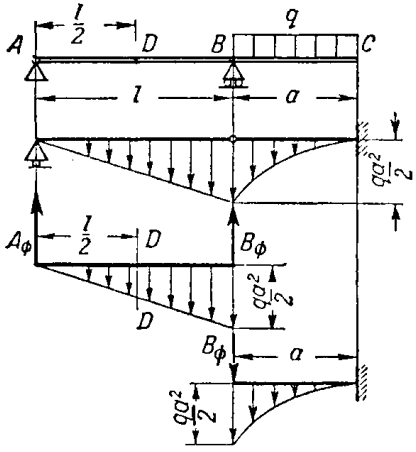
Solution. We draw the bending moment diagram. We consider this diagram as a fictitious continuous load, directed downwards (since $M < 0$), and the zero line of the diagram as the axis of a fictitious beam, the constraints at the supports of which are chosen in accordance with the end conditions of the given actual beam. The hinge introduced at point B of the fictitious beam divides it into two beams: the suspended beam AB, and the cantilever BC. We calculate the "load" areas ω_1 and ω_2 of the M diagram, and the fictitious reactions A_Φ and B_Φ of the suspended beam:

$$\omega_1 = \frac{1}{2}\,\frac{qa^2}{2}\,l = \frac{qa^2}{4}\,l;$$

$$\omega_2 = \frac{1}{3}\,\frac{qa^2}{2}\,a = \frac{qa^3}{6};$$

$$A_\Phi = \frac{1}{3}\,\omega_1 = \frac{qa^2}{12}\,l;$$

$$B_\Phi = \frac{2}{3}\,\omega_1 = \frac{qa^2}{6}\,l.$$



PROBLEM 5.34

The bending moment at section D, due to the fictitious load, is

$$M_D^\Phi = \left(A_\Phi \times \frac{l}{2}\right) - \left(\omega_D \times \frac{1}{3} \times \frac{l}{2}\right),$$

where ω_D is the "load" area lying on the left of the section DD, and equalling

$$\omega_D = \frac{1}{2} \times \frac{qa^2}{4} \times \frac{l}{2} = \frac{qa^2 l}{16}.$$

Consequently

$$M_D^{\Phi} = \left(\frac{qa^2 l}{12} \times \frac{l}{2}\right) - \left(\frac{qa^2 l}{16} \times \frac{l}{6}\right) = \frac{qa^2 l^2}{32},$$

and

$$f_D = \frac{M_D^{\Phi}}{EJ} = \frac{qa^2 l^2}{32EJ}.$$

To determine the deflection of the end section C of the beam, we consider the fictitious beam BC separately. The action of the suspended beam AB resting on it is replaced by a thrust B_{Φ} transmitted through the hinge. Then

$$M_C^{\Phi} = -B_{\Phi} a - \left(\omega_2 \times \frac{3}{4} \times a\right) = -\frac{qa^2 l}{6} a - \left(\frac{qa^3}{6} \times \frac{3}{4} a\right)$$

$$= -\frac{qa^3}{24}(4l + 3a);$$

$$f_C = -\frac{qa^3}{24EJ}(4l + 3a).$$

The slopes at supports A and B of the beam are determined from the formula $\theta = Q_{\Phi}/EJ$, where $Q_A^{\Phi} = A_{\Phi}$, and $Q_B^{\Phi} = -B_{\Phi}$:

$$\theta_A = \frac{A_{\Phi}}{EJ} = \frac{qa^2 l}{12EJ}; \qquad \theta_B = -\frac{B_{\Phi}}{EJ} = -\frac{6EJ}{qa^2 l}.$$

5.35. Applying the semi-graphical method determine the value of the deflection at the mid-span, and the slopes at the supports of the beams shown in the figure.

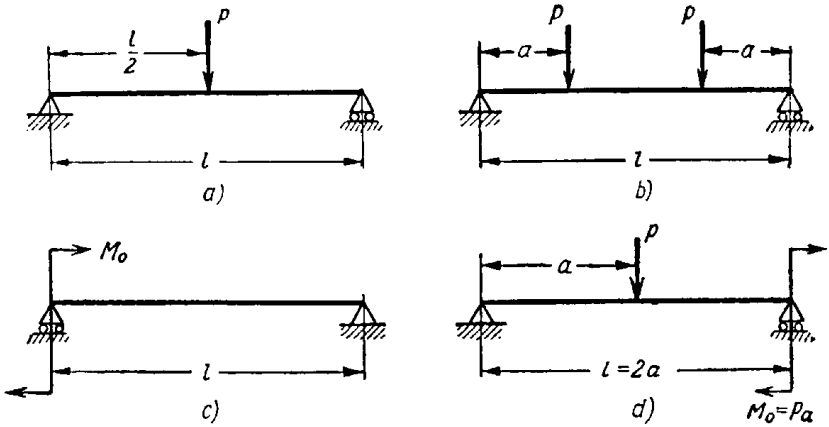
Answer: (a) $f = -\dfrac{Pl^3}{48EJ}$, $\theta_A = -\theta_B = -\dfrac{Pl^2}{16EJ}$;

(b) $f = \dfrac{Pa}{24EJ}(4a^2 - 3l^2)$,

$\theta_A = -\theta_B = -\dfrac{Pa^2}{2EJ}(l - a)$;

(c) $f = -\dfrac{M_0 l^2}{16EJ}$, $\theta_A = -\dfrac{M_0 l}{3EJ}$, $\theta_B = \dfrac{M_0 l}{6EJ}$;

(d) $f = \dfrac{Pa^3}{13EJ}$, $\theta_A = \dfrac{Pa^2}{12EJ}$, $\theta_B = -\dfrac{5}{12}\dfrac{Pa^2}{EJ}$.



PROBLEM 5.35

5.36. A timber beam of square cross-section is simply supported and loaded by a concentrated force at the mid-span. The maximum stresses must not exceed 80 kg/cm², and the maximum deflection at the mid-span must not exceed 1/400 of the span. Determine the necessary depth of the beam, if the span is 6 m.

Answer: $h = 32$ cm.

5.37. A simply supported steel No. 22a I-beam, having a span of 5 m, is loaded by a concentrated force of 6 t at a distance of 1·3 m from the left-hand support. Determine whether the following conditions are fulfilled:

(a) the maximum normal stress must not exceed 1900 kg/cm²;

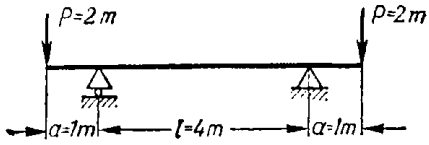
(b) the maximum deflection must not exceed 1/360 of the span.

Answer: $\sigma = 1870$ kg/cm²; $f = 2{\cdot}3$ cm $> l/360$.

5.38. Find, semi-graphically, the deflections at the mid-span, and at the overhung ends of the beam shown in the figure. Check the value of the deflection at the mid-span by means of a graphical construction. $EJ = 2 \times 10^9\ \mathrm{kg\,cm^2}$.

Answer: $f_1 = \dfrac{Pal^2}{8EJ} = 2\ \mathrm{cm}$;

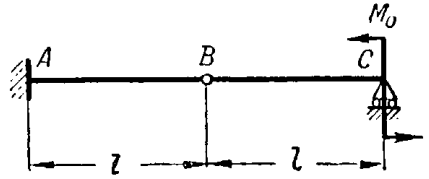
$$f_2 = -\frac{Pa^2}{6EJ}(3l + 2a) = -2{\cdot}33\ \mathrm{cm}.$$



PROBLEM 5.38

5.39. Find the deflection of point B of the beam shown in the figure.

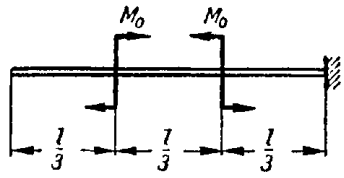
Answer: $f = -\dfrac{M_0 l^2}{3EJ}$.



PROBLEM 5.39

5.40. Find graphically, and also semi-graphically, the deflection at the mid-span of the beam shown in the figure, if $M_0 = 3\ \mathrm{tm}$, $l = 6\ \mathrm{m}$, $EJ = 3 \times 10^9\ \mathrm{kg\,cm^2}$.

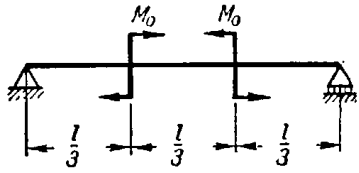
Answer: $f = -\dfrac{5}{72}\dfrac{M_0 l^2}{EJ} = 2{\cdot}5\ \mathrm{cm}$.



PROBLEM 5.40

5.41. Find graphically, and also semi-graphically, the deflection of the free end of the cantilever shown in the figure, if $M_0 = 6$ tm, $l = 3$ m and $EJ = 3 \times 10^9$ kg cm^2.

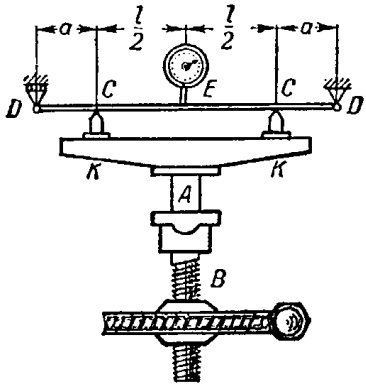
Answer: $f = \dfrac{M_0 l^2}{6EJ} = 3$ cm.



PROBLEM 5.41

5.42. In testing machines of the Korobov type the force transmitted by screw B to test specimen A is measured by the value of the deflection of steel beam DD (see figure). The thrust to the beam DD is transmitted, at points C, by means of a rigid transverse member KK. The dimensions of the beam and its cross-section are: DD = 1 m, $a = 0{\cdot}1$ m; $b = 6$ cm, $h = 4$ cm. Establish the value of the load P applied to the test-piece, if the reading of the deflection gauge E increases by 1 mm. At what distance (a) must the support prisms be placed on the transverse member, so that a deflection of 1 mm of beam DD corresponds to a load, P, of 500 kg?

Answer: $P = 1040$ kg; $a = 23$ cm.



PROBLEM 5.42

5.43. A simply supported steel No. 24a I-beam, having a span of 5 m, is loaded by a force P at a section lying at one quarter-span from the left-hand support. Determine the maximum deflection of the beam, if the allowable stress is 1400 kg/cm^2.

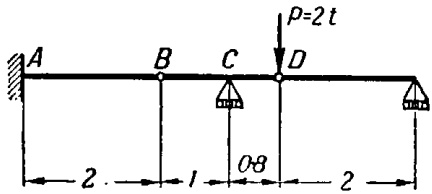
Answer: $f = 1{\cdot}13$ cm.

5.44. A simply supported steel No. 30a I-beam is loaded by a force P applied to the mid-span. By means of extensometers placed at the quarter-span, on the level of the neutral layer, the magnitude of the principal stresses is measured at the axis of the beam. It was found that $\sigma_1 = +400$ kg/cm^2 $= -\sigma_3$. Determine the value of the slope at the supports, if the span l is 2·0 m. Find also the value of the maximum normal stress in the critical section of the beam.

Answer: $\theta = \dfrac{\sigma_1 l^2}{8ES}\delta = 0{\cdot}0026$; $\sigma_{max} = 1550$ kg/cm^2, where S is the first moment of area of one half of the cross-section, and δ is the thickness of the web.

5.45. Using the semi-graphical method find the deflection at point D, and the slope at point C of the beam shown in the figure. The flexural rigidity EJ of the beam is 5×10^9 kg cm^2. The dimensions in the figure are given in metres.

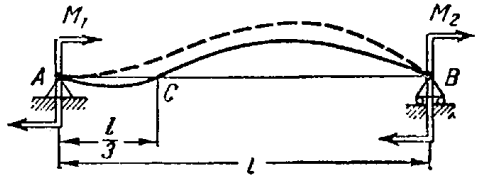
Answer: $f_D = -0{\cdot}85$ cm; $\theta_C = -0{\cdot}0096$.



PROBLEM 5.45

5.46. A simply supported beam AB is bent by couples applied to its ends (see figure). What is the ratio of the values of these couples, if the point of inflection of the elastic curve has been found to be at $\dfrac{1}{3}\,l$ from the left-hand support? What is the slope of the elastic curve at point C, and at support A? Specify the shape of the elastic curve of the beam (whether it is shown by the continuous line or by the broken line in the figure).

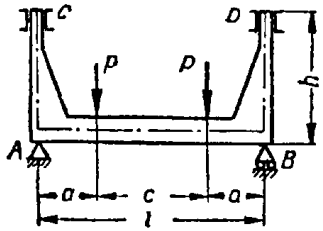
Answer: $M_2 = 2M_1$; $\theta_C = \dfrac{M_1 l}{6EJ}$; $\theta_A = 0$.



PROBLEM 5.46

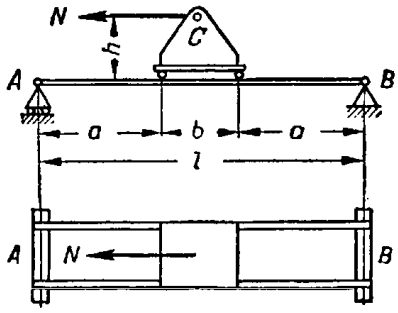
5.47. A transverse beam AB of an open bridge is bent by symmetrically placed loads P (see figure). Find the amount by which the top flanks C and D approach each other, if the flexural rigidity of the transverse beam is constant, and equal to EJ.

Answer: $\delta = \dfrac{Pah}{2EJ}(a + c)$.



PROBLEM 5.47

5.48. A hand winch is fixed to two wooden beams of circular cross-section, with $d = 24$ cm, as shown diagrammatically in the



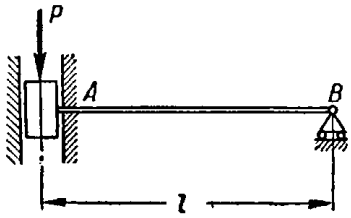
PROBLEM 5.48

figure. Determine the value of the horizontal displacement of point C of the winch due to a horizontal tension $N = 4$ t, in the cable, if $l = 6$ m, $b = 1$ m, $h = 1$ m. The longitudinal force set up in the beams is to be neglected.

Answer: $\delta = 3{\cdot}74$ mm.

5.49. A beam, having at one end a rigidly attached slider which freely (without friction) slides along vertical guides, is supported by its other end on a movable hinged support (see figure). Determine the value of the displacement of the slider, if a vertical force P is applied to it. What should be the end constraints of the fictitious beam?

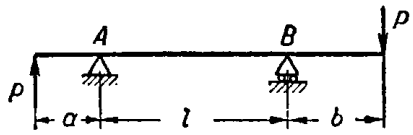
Answer: $f = -\dfrac{Pl^3}{3EJ}$.



PROBLEM 5.49

5.50. A beam, having a span l and two overhangs a and b, is loaded by two equal forces applied to its ends. What must be the length of the overhang b, if no rotation of section is to occur at support A?

Answer: $b = 2a$.

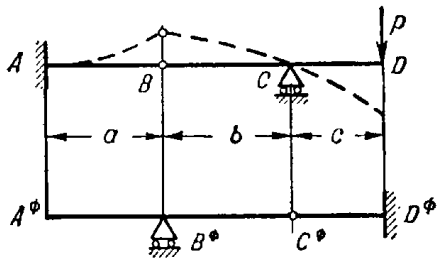


PROBLEM 5.50

5.51. At what distance from the constraint must hinge B be located, so that the supported beam AB and the suspended beam BC have a common tangent at point B?

Instruction. In order to secure the equality of the angles $\theta_{B,a}$ and $\theta_{B,c}$, it will be necessary to equate to zero the fictitious reaction at point B.

Answer: $a = \dfrac{b}{2}$.

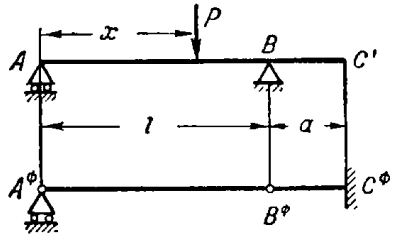


PROBLEM 5.51

5.52. A beam, with a span l and overhang a, freely supported at points A and B, is bent by a concentrated force P. At what distance x from the support A must the force P be applied, in order to obtain the maximum deflection of the free end C of the beam (see figure).

Instruction. Express the fictitious reaction at point B in the form of a function of the distance x, and investigate it for a maximum. Max f_C will be determined as dependent on max B_Φ.

Answer: $x = \dfrac{1}{\sqrt{3}} \approx 0{\cdot}577\,l$.

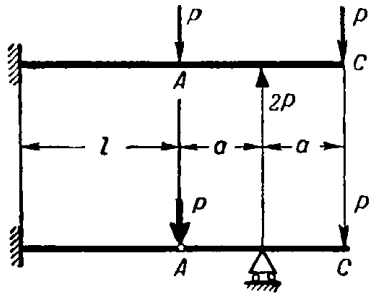


PROBLEM 5.52

5.53. Which of the two beams shown in the figure has larger angular and linear displacements at points A and C? Determine the magnitudes of these displacements.

Answer: (a) $\theta_A = 0,\ f_A = 0,\ \theta_C = -\dfrac{Pa^2}{EJ},\ f_C = -\dfrac{Pa^3}{EJ}$;

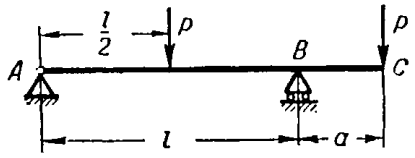
(b) $\theta_A = \dfrac{Pa^2}{6EJ},\ f_A = 0,\ \theta_C = -\dfrac{5Pa^2}{6EJ},\ f_C = -\dfrac{2Pa^3}{EJ}$.



PROBLEM 5.53

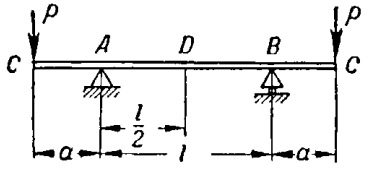
5.54. A beam ABC on two supports, with an overhang, is bent by two equal forces P applied to the mid-span and the free end respectively (see figure). For what length a of overhang will the deflection be zero at point C?

Answer: $a = 0{\cdot}16l$.



PROBLEM 5.54

5.55. (a) For what length of overhangs will the deflections at points C and D of the beam, shown in the figure, be equal in absolute value? (b) At what length of overhangs will the maximum



PROBLEM 5.55

(in absolute value) deflection be at mid-span? The total length of the beam $L = l + 2a$ remains the same in both cases.

Answer: (a) $a = 0{\cdot}152L$; (b) $a = 0{\cdot}167L$.

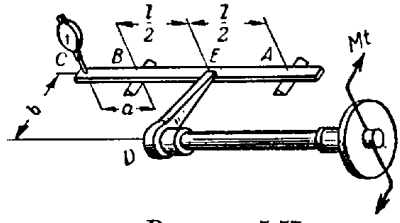
5.56. A beam of uniform cross-section rests on two supports equidistant from the respective ends. Determine the distance a between the supports as a function of the length l of the beam, for which the maximum deflection, due to the weight of the beam has the least value.

Answer: $a = 0{\cdot}52l$.

5.57. In one of the machines for torsion testing the value of the torque being transmitted by a shaft is measured as the value of the deflection of a steel beam AB with an overhang $BC = a$. In the figure the diagram of the force measuring arrangement is shown, in which the shaft being tested works together with a clamp and a rigid lever DE as a whole. When the shaft is twisted the lever DE transmits the thrust to beam AB at point E (the mid-span).

What is the value of the torque M applied to the shaft, which will change the reading of the deflection gauge (mounted at point C) by 1 mm. The length b of the lever is 20 cm, the span l of the beam is 60 cm; a is 10 cm, and the dimensions of the beam cross-section are 35×10 mm²?

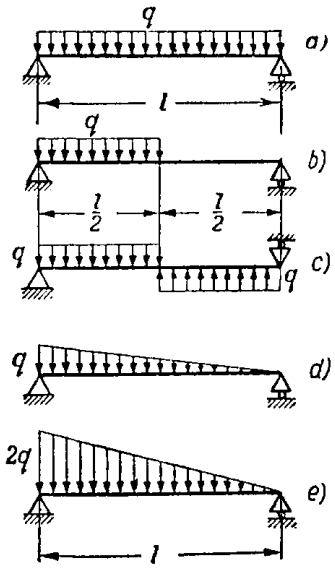
Answer: $M_t = 5{\cdot}2$ kgm.



PROBLEM 5.57

5.58. Applying the method of superposition compare the deflections at mid-span of the identical beams shown in the figure. Taking the deflection of the beam (a) as unity, calculate the deflection of mid-span of the remaining four beams. Establish how many times the maximum deflection of beam (a) is greater than the maximum deflection of beam (c).

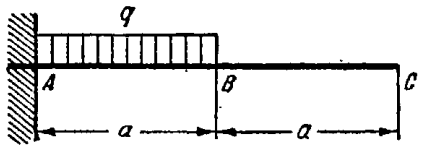
Answer: 16 times.



PROBLEM 5.58

5.59. Determine how the value of the angular displacement, and that of the linear displacement of the end section of the cantilever shown in the figure will change, if the continuous load, uniformly distributed along the length AB = a, is transferred to portion BC.

Answer: The slope increases 7 times, and the deflection 5·85 times.



PROBLEM 5.59

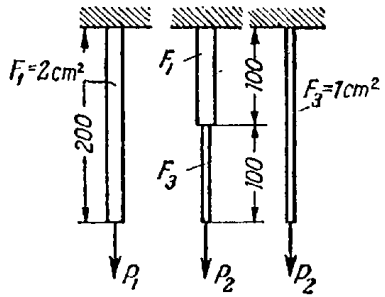
5.60. A steel bar, length l, is lifted by a crane using a rope fastened to the middle of the bar. What will be the maximum normal stress, and the deflection of the ends of the beam due to its own weight, if flexure takes place: (a) in the plane of the maximum rigidity, (b) in the plane of the minimum rigidity? The cross-section of the bar is I No. 40a. Its length l is 10 m.

Answer: (a) $\sigma = 78$ kg/cm², $f = 0{\cdot}12$ cm;
(b) $\sigma = 900$ kg/cm², $f = 4{\cdot}0$ cm.

22. ENERGY METHODS OF DETERMINING DEFORMATIONS

5.61. Compare the quantities of strain energy stored in the three steel bars shown in the figure, if the stress in the lower portion of the second bar, and in the first and the third bar is 2000 kg/cm².

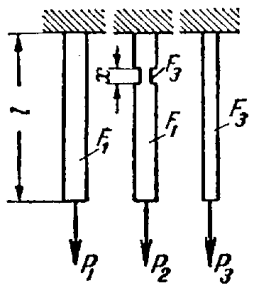
Answer: 8:3:4.



PROBLEM 5.61

5.62. Compare the quantities of strain energy stored in three bars (see figure) made of steel, with the condition that the maximum stresses do not exceed the yield point $\sigma_y = 2000$ kg/cm². The axial dimension x of the cut in the second bar is negligible. $F_1 = 2$ cm²; $F_3 = 1$ cm²; $l = 100$ cm; $E = 2 \times 10^6$ kg/cm².

Answer: 200 kg cm; 50 kg cm; 100 kg cm.



PROBLEM 5.62

5.63. Determine the maximum value of the specific (elastic) strain energy of bars subjected to tension, if they are made of the following materials:

Material	E kg/cm²	σ_p kg/cm²	*Answer:* a kg cm/cm³
Grey cast iron (tension)	1×10^6	500	0·125
Grey cast iron (compression)	$1{\cdot}1 \times 10^6$	1000	0·455
Mild steel	2×10^6	2000	1·000
Oak	$1{\cdot}2 \times 10^5$	250	0·260
Pine	$0{\cdot}9 \times 10^5$	150	0·125
Rubber	80	20	2·500

5.64. Two circular bars with diameters d_1 and d_2 have the same length. They are made of the same material and subjected to equal torques. What is the relation between the quantities of strain energy stored in the two bars?

Answer: $U_1 : U_2 = d_2^4 : d_1^4$.

5.65. A shaft transmitting 100 h.p. at 120 rev/min is twisted through 1° over a length of 3·6 m. Find the strain energy stored per 1 m of the shaft length.

Answer: 1·45 kg m.

5.66. A shaft transmits 100 h.p. at 120 rev/min. Determine the strain energy stored per 1 m of the shaft length, if the maximum shear stress is 350 kg/cm², and the modulus of rigidity is 8×10^5 kg/cm².

Answer: 274 kg cm.

5.67. A solid shaft of circular cross-section 10 cm in diameter is subjected to a constant bending moment of 34,500 kg cm, and simultaneously to a twisting moment of 46,000 kg cm. Determine the strain energy stored per 1 m of the shaft length.

$$E = 2 \times 10^6 \text{ kg/cm}^2; \quad G = 8 \times 10^5 \text{ kg/cm}^2.$$

Answer: 195 kg cm approx.

5.68. Two wires, one of steel, the other of bronze, have the same length and the same diameter. Two identical helical springs are made out of them. The yield point of steel is four times the yield point of bronze. The modulus of elasticity of steel is twice the modulus of elasticity of bronze. Determine the ratio of the maximum amounts of strain energy stored in the springs. Find the ratio of the safe loads for the springs.

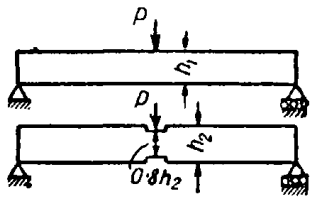
Answer: 8 : 1 and 4 : 1.

5.69. A simply supported beam of rectangular cross-section $12 \times 20\ \text{cm}^2$ is loaded over the whole length of span $l = 2$ m by a uniformly distributed load $q = 1{\cdot}2$ t/m. The modulus of elasticity E of the beam (pine) is $1 \times 10^5\ \text{kg/cm}^2$. Calculate the amount of strain energy stored. By how much will this amount change if the beam is supported on its longer side?

Answer: $U = \dfrac{q^2 l^5}{240 EJ} = 240$ kg cm. It increases to 667 kg cm.

5.70. Compare the amount of strain energy stored in two beams (shown in the figure) which are tested in flexure, and have rectangular cross-sections of the same width. The maximum stress in each beam, due to forces P, is brought up to the yield point. The first beam has a constant depth along the whole length; the second beam has a cut at the point of application of the force P, reducing the depth by one-fifth. The axial length of the cut in the beam is negligible.

Answer: $U_2 : U_1 = 0{\cdot}512$.



PROBLEM 5.70

5.71. A steel beam of circular cross-section, 5 cm in diameter, having a span of 75 cm, is simply supported and loaded at the

mid-span by a concentrated force. The maximum normal stress $[\sigma]$ is 1000 kg/cm^2. Find the amount of strain energy stored in the beam.

Answer: $\frac{2}{3}\frac{[\sigma]^2 Jl}{Ed^2} = 30{\cdot}7$ kg cm.

5.72. A beam of rectangular cross-section is subjected to the action of a constant bending moment. The maximum normal stress is σ, the modulus of elasticity is E, the cross-sectional area is F and the length of the beam is l. Calculate the amount of strain energy stored.

Answer: $\frac{\sigma^2 Fl}{6E}$.

5.73. An I-beam (No. 20a), 6 m in span, is supported at one end on a fixed hinge and at the other end on a helical spring 10 cm in diameter. The diameter of the rod from which the spring is made is 2 cm; there are ten coils. The beam and the spring are of steel having a modulus of elasticity, E, of 2×10^6 kg/cm^2, and a modulus, G, of 8×10^5 kg/cm^2. Determine the amount of strain energy stored in this beam when loaded by a vertical force of 2 t applied to the mid-span of the beam.

Solution. We shall refer to the spring deflection as Δl, and to the value (measured from the straight horizontal axis of the beam) of the beam deflection, due to force P, as f. The amount of strain energy stored in the beam is expressed as

$$U = A = \frac{1}{2} P \left(f + \frac{1}{2}\Delta l\right)$$

$$= \frac{P^2 l^2}{96EJ} + \frac{P^2 R^3 n}{2Gr^4} = \frac{P^2}{10^6}\left(\frac{600^3}{96 \times 2 \times 2370} + \frac{5^3 \times 10}{2 \times 0{\cdot}8 \times 1^4}\right)$$

$$= 4(475 + 782) = 5028 \text{ kg cm}.$$

Answer: $U = 5028$ kg cm.

5.74. A carbon steel beam, with one end built into a wall, is loaded at the free end by a force giving rise to a maximum normal stress equal to the limit of proportionality (2000 kg/cm^2). The beam has

a circular cross-section 25 mm in diameter, and a length of 120 cm. What will be the required diameter, if this beam is made of a nickel steel? The amount of energy stored is not to be altered, and the maximum normal stress should also equal the limit of proportionality (3500 kg/cm²).

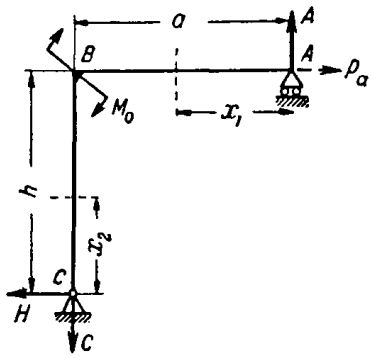
Which beam will support a greater static load when loaded up to the limit of proportionality? Take the moduli of elasticity for the materials of both beams as equal.

Answer: $d_2 = 1{\cdot}43$ cm; the first beam will support a load 3·96 times greater.

5.75. A bar of circular cross-section, with a diameter d and a length l, is loaded by a load P in the following two ways: (a) applied to give it an axial pull; (b) applied transversely at the mid-point while its ends are simply supported. Compare the strain energies of the bar for both cases, and show that the specific energy of any bar, loaded arbitrarily, is proportional to the square of the stress.

Answer: $1 : \frac{l^2}{3d^2}$; $\quad a_a = \frac{\sigma^2}{2E}$; $\quad a_b = \frac{\sigma^2_{\max}}{24E}$.

5.76. Applying Castigliano's theorem, find the horizontal displacement of point A of the structure shown in the figure.



PROBLEM 5.76

Solution. To find the horizontal displacement of point A it will be necessary to differentiate the energy (due to strain) of the

system with respect to an additional horizontal force applied to point A. The horizontal displacement of point A,

$$\Delta_A = \int \frac{M_1 \, dx}{EJ} \frac{\partial M_2}{\partial P_a} + \int \frac{M_1 \, dx}{EJ} \frac{\partial M_2}{\partial P_a}.$$

Using the co-ordinates adopted in the figure, the limits of the first integral will be 0 and a, and those of the second integral will be 0 and h. We shall determine the reactions due to the given couple M and the additional force P_a:

$$H = P_a, \quad A = C = \frac{M_0 + P_a h}{a}.$$

We shall then calculate the bending moments M_1 and M_2, and their derivatives with respect to P_a:

$$M_1 = A x_1 = \frac{M_0 + P_a h}{a} x_1; \qquad \frac{\partial M_1}{\partial P_a} = + \frac{h}{a} x_1;$$

$$M_2 = H x_2 = P_a x_2; \qquad \frac{\partial M_2}{\partial P_a} = + x_2.$$

The values obtained will be substituted in the formula for Δ_A:

$$\Delta_A = \frac{1}{EJ} \int_0^a (M_0 + P_a h) \frac{x_1}{a} \frac{h}{a} x_1 \, dx + \frac{1}{EJ} \int_0^h P_a x_2 x_2 \, dx.$$

As $P_a = 0$, the second integral is zero, and

$$\Delta_A = \frac{1}{EJ} \int_0^a M_0 \frac{h}{a_2} x_1^2 \, dx = \frac{M_0 h a}{3EJ}.$$

5.77. Using the Maxwell–Mohr method, determine the deflection of end A of the left-hand cantilever of the frame ABCD shown in figure (a), subjected to a load P at the end of the other cantilever.

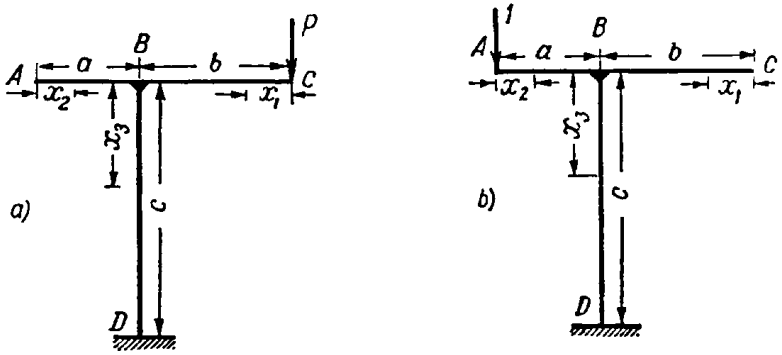
Solution. A load $P^0 = 1$ is applied to the section at which the deflection is being determined (see figure (b)).

We shall write down the expressions of the bending moment, due to the forces P and P^0, for each of the three portions of the frame:

$$M_1 = -Px; \qquad M_1^0 = 0;$$

$$M_2 = 0; \qquad M_2^0 = -x;$$

$$M_3 = -Pb; \qquad M_3^0 = a.$$



PROBLEM 5.77

The required displacement,

$$f_A = \frac{1}{EJ} \sum \int_l M M^0 \, dx.$$

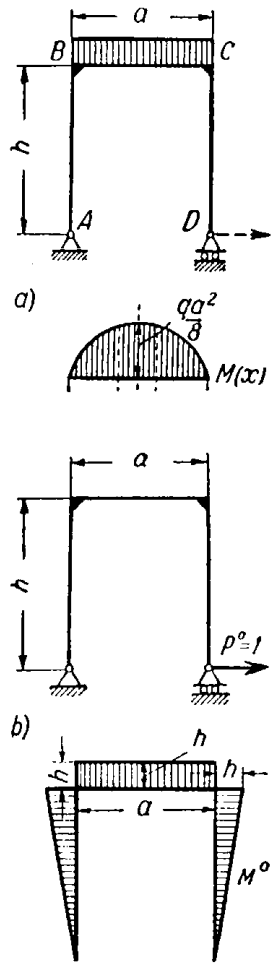
Since in the first portion $M_1^0 = 0$, and in the second portion $M_2 = 0$, the only integral to be calculated is the one for the third portion:

$$f_A = \frac{1}{EJ} \int_0^c -Pba \, dx = -\frac{Pabc}{EJ}.$$

The minus sign indicates that end A is lifted up (in the opposite direction to the force $P^0 = 1$).

5.78. Applying Vereshchagin's method, determine the displacement of hinge D of the symmetrical frame shown in the figure. The frame is loaded by a uniformly distributed load on the hori-

zontal span. The cross-sections of the legs and the horizontal span of the frame are identical.



PROBLEM 5.78

Solution. Any displacement is given as $\Delta = \omega M_C^0 / EJ$, and when loaded in several portions, as

$$\Delta = \sum \frac{\omega M_C^0}{EJ}.$$

Here ω is the area of $M(x)$ diagram due to the given load (figure (a)); M_C^0 is the ordinate of the bending moment diagram due to the load $P^0 = 1$ applied to the point whose deflection is being determined (figure (b)). The ordinate M_C^0 is taken at the point in the section of the beam (or the frame) which contains the centroid of the $M(x)$ diagram of the given load.

We shall consider only the horizontal portion, since $M(x)$ in the legs of the frame is zero:

$$\omega M_C = \left[\frac{2}{3}\frac{qa^2}{8}a\right]h = \frac{qa^3h}{12}.$$

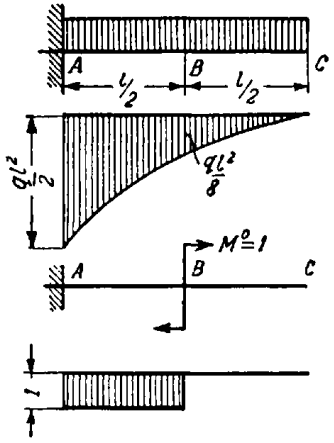
The displacement of support D is

$$\Delta_D = \frac{qa^3h}{12EJ}.$$

5.79. Determine the slope of section B of the beam shown in the figure. It is loaded by a uniformly distributed load q. Apply Vereshchagin's method.

Solution. The area of the bending moment diagram

$$\omega = -\left(\frac{1}{3} \times \frac{ql^2}{2} \times l\right) + \left(\frac{1}{3} \times \frac{ql^2}{8} \times \frac{l}{2}\right) = -\frac{7ql^3}{48}.$$



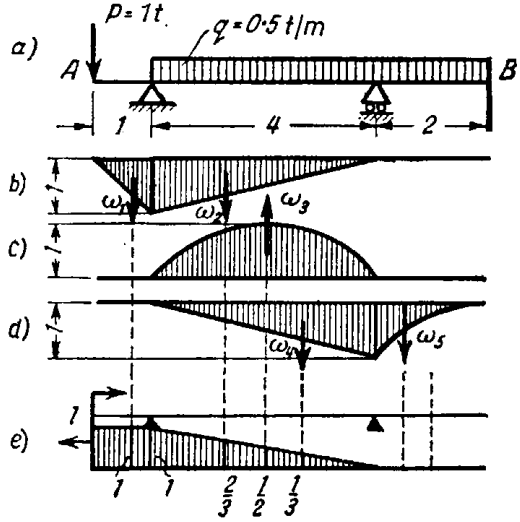
PROBLEM 5.79

The ordinate of a unit load of moment $M^0 = 1$, applied to section B, is taken at a point in line with the centroid of the M diagram: $M^0_C = -1$. The required slope is

$$\theta_B = +\frac{7}{48}\frac{ql^3}{EJ}.$$

The plus sign indicates that the direction of rotation of section B coincides with the direction of M^0.

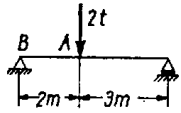
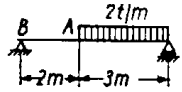
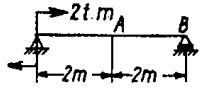
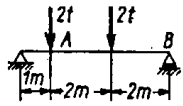
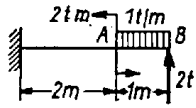
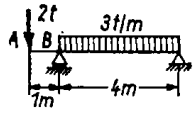
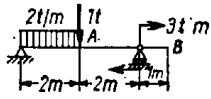
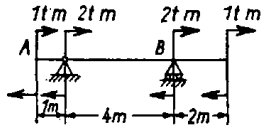
5.80. Determine the slope at the end of the left-hand overhang of beam AB loaded as shown in the figure. Make use of Vereshchagin's method. $E = 2 \times 10^6$ kg/cm^2; $J = 2000$ cm^4.



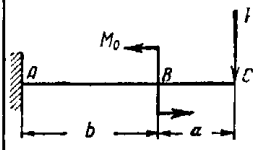
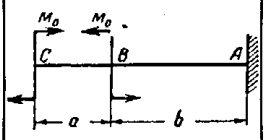
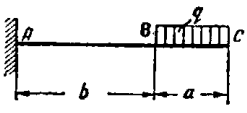
PROBLEM 5.80

Solution. We make use of the principle of superposition, and draw separate diagrams for the force P (diagram b) the load q within the limits of the span (diagram c), the load q on the overhang (diagram d).

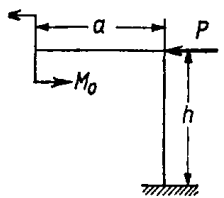
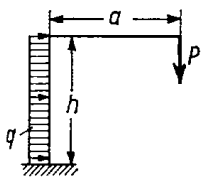
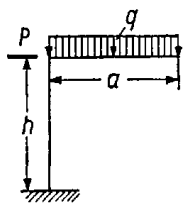
We calculate the areas of these diagrams:

	Diagram of beam	J cm^4	*Answer*	
			f_A, cm	θ_B
a		1500	−1·6	−0·0107
b		4000	−1·237	−0·007 69
c		2000	−0·5	+0·003 33
d		2500	−0·96	+0·0096
e		3000	+1·611	+0·0164
f		3500	+0·667	−0·0076
g		2500	−0·333	−0·00133
h		2500	+0·5	−0·04

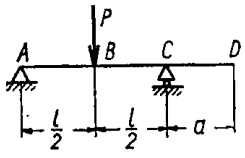
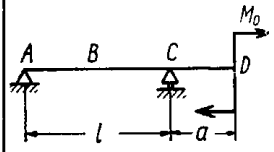
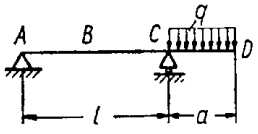
PROBLEM 5.81

	Diagram of beam	*Answer*
a		$\theta_B = -\frac{Pb(2a+b)}{2EJ} + \frac{M_0 b}{EJ} = -0{\cdot}0094$ $\theta_C = -\frac{Pl^2}{2EJ} + \frac{M_0 b}{EJ} = -0{\cdot}0135$ $f_B = -\frac{Pb^2}{6EJ}(3a+2b) + \frac{M_0 b^2}{2EJ} = -1{\cdot}88$ cm $f_C = -\frac{Pl^3}{3EJ} + \frac{M_0 b}{2EJ}(2a+b) = -4{\cdot}3$ cm
b		$\theta_B = 0;\quad \theta_C = -\frac{M_0 a}{EJ} = -0{\cdot}0083$ $f_B = 0;\quad f_C = -\frac{M_0 a^2}{2EJ} = 0{\cdot}83$ cm
c		$\theta_B = -\frac{qabl}{2EJ} = -0{\cdot}031\,25$ $\theta_C = -\frac{qa}{6EJ}(a^2+3ab+3b^2) = -0{\cdot}034$ $f_B = -\frac{qab^2}{12EJ}(3a+4b) = -5{\cdot}6$ cm $f_C = -\frac{qa}{24EJ}(3a^3+12a^2b+18ab^2+8b^3)$ $= -12{\cdot}3$ cm

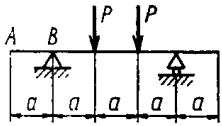
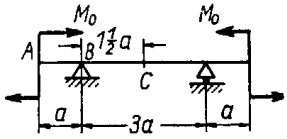
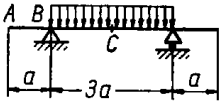
PROBLEM 5.82

	Diagram of frame	Numerical data	*Answer*
a		$P = 2$ t $M_0 = 1$ t m $a = 2$ m $h = 4$ m $J = 10^4$ cm^4	$f = \frac{1}{EJ}\left[M_0 \frac{a^2}{2} + M_0 ah + \frac{Pah^2}{2}\right] = 2{\cdot}1$ cm $\Delta = \frac{1}{EJ}\left[M_0 \frac{h^2}{2} + \frac{Ph^3}{3}\right] = 2{\cdot}53$ cm $\theta = \frac{1}{EJ}\left[M_0(a+h) + \frac{Ph^2}{2}\right] = 0{\cdot}0111$
b		$P = 2$ t $q = 1$ t/m $a = 1$ m $h = 4$ m $J = 8000$ cm^4	$f = \frac{1}{EJ}\left[\frac{Pa^3}{3} + Pa^2h + \frac{qah^3}{6}\right] = 1{\cdot}21$ cm $\Delta = \frac{1}{EJ}\left[\frac{Pah^2}{2} + \frac{qh^4}{8}\right] = 3$ cm $\theta = \frac{1}{EJ}\left[\frac{Pa^2}{2} + Pah + \frac{qh^3}{6}\right] = 0{\cdot}0123$
c		$P = 2$ t $q = 1$ t/m $a = 1$ m $h = 3$ m $J = 4000$ cm^4	$f = \frac{1}{EJ}\left[\frac{qa^4}{8} + \frac{qa^3h}{2} + \frac{Pah^2}{2}\right] = 1{\cdot}33$ cm $\Delta = \frac{1}{EJ}\left[\frac{qa^2h^2}{4} + \frac{Ph^3}{3}\right] = 2{\cdot}53$ cm $\theta = \frac{1}{EJ}\left[\frac{qa^3}{6} + \frac{qa^2h}{2} + \frac{Ph^2}{2}\right] = 0{\cdot}0133$

PROBLEM 5.83

	Diagram of beam and loading	*Answer*					
		Deflections cm		Slopes			
		B	D	A	B	C	D
a		− 0·33	+ 0·25	− 0·0025	0	0·0025	0·0025
b		1·00	− 1·83	0·006 67	0·001 67	− 0·013 33	− 0·023 33
c		0·25	− 0·396	0·001 67	0·000 42	0·003 33	− 0·004 17

PROBLEM 5.84

	Loading Diagram	*Answer*	
		Deformation at end of overhang A	Slope at support; deflection at mid-span
a		$\theta = -\frac{Pa^2}{EJ} = -0{\cdot}00417$ $f = +\frac{Pa^3}{EJ} = +0{\cdot}42$ cm	$\theta = -\frac{Pa^2}{EJ} = -0{\cdot}00417$ $f = -\frac{23}{24}\frac{Pa^3}{EJ} = -0{\cdot}4$ cm
b		$\theta = -\frac{5M_0a}{2EJ} = -0{\cdot}0125$ $f = +\frac{2M_0a^2}{EJ} = +1{\cdot}67$ cm	$\theta = -\frac{3M_0a^2}{2EJ} = -0{\cdot}0208$ $f = -\frac{9}{8}\frac{M_0a^2}{EJ} = -0{\cdot}94$ cm
c		$\theta = -\frac{9}{8}\frac{qa^3}{EJ} = -0{\cdot}0047$ $f = +\frac{9}{8}\frac{qa^4}{EJ} = +0{\cdot}47$ cm	$\theta = -\frac{9}{8}\frac{qa^3}{EJ} = -0{\cdot}0047$ $f = -\frac{135}{128}\frac{qa^4}{EJ} = -0{\cdot}44$ cm

PROBLEM 5.85

$$\omega_1 = \frac{1}{2} \times 1 \times 1 = 0.5\ \text{tm}^2; \qquad \omega_2 = \frac{1}{2} \times 1 \times 4 = 2\ \text{tm}^2;$$

$$\omega_3 = \frac{2}{3} \times 1 \times 4 = \frac{8}{3}\ \text{tm}^2; \qquad \omega_4 = \frac{1}{2} \times 1 \times 4 = 2\ \text{tm}^2;$$

$$\omega_5 = \frac{1}{3} \times 1 \times 2 = \frac{2}{3}\ \text{tm}^2;$$

We apply the couple $M^0 = 1$ to the beam at the end of the left-hand overhang and plot the M^0 diagram for this loading (diagram f). We calculate (and mark on the diagram) the values of the ordinates M^0_C, the position of which coincides with the centroids of the areas (calculated earlier) of the bending moment diagram.

We carry out the following calculations:

$$EJ\theta_A = -(0.5 \times 1) - \left(2 \times \frac{2}{3}\right) + \left(\frac{8}{3} \times \frac{1}{2}\right) - \left(2 \times \frac{1}{3}\right)$$

$$= -\frac{1}{2} - \frac{4}{3} + \frac{4}{3} - \frac{2}{3} = -\frac{7}{6}\ \text{tm}^2;$$

$$\theta_A = -\frac{7 \times 10}{6 \times 2 \times 10^6 \times 2 \times 10^3} = -\frac{7}{2400}.$$

5.81. Determine the deflection at point A, and the slope at point B for the beams given diagrammatically in the table on page 261; $E = 2 \times 10^6\ \text{kg/cm}^2$.

5.82. A cantilever ABC of uniform cross-section is loaded as shown diagrammatically in the table on page 262. Determine the deflections and the slopes at sections B and C. $P = 2$ t; $M_0 = 4$ tm; $q = 2$ t/m; $b = 3$ m; $a = 2$ m; $E = 2 \times 10^6\ \text{kg/cm}^2$; $J = 4800\ \text{cm}^4$; $l = a + b = 5$ m.

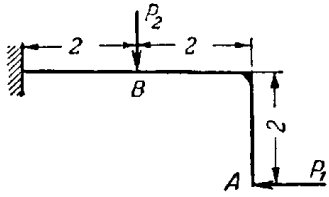
5.83. Determine the vertical displacement f, the horizontal displacement Δ, and the slope θ at the free end of the steel ($E = 2 \times 10^6\ \text{kg/cm}^2$) frames of uniform cross-section, represented in the loading diagrams a, b, c on page 263.

5.84. The loading diagrams on page 264 refer to a steel ($E = 2 \times 10^6 \text{kg/cm}^2$) beam of uniform cross-section ($J = 2000 \text{cm}^4$), having a span, l, of 4 m and a single overhang a of 1 m. Determine the deflection at sections B and D, and the slope at sections A, B, C, and D due to the following loads: $P = 1$ t; $q = 2$ t/m; $M_0 = 4$ tm.

5.85. The loading diagrams given on page 265 refer to a symmetrical double steel beam of uniform cross-section overhanging the supports at both ends. Determine the deflection at the mid-span and at the ends of the overhangs. Determine also the rotation of the overhang A on support B. $E = 2 \times 10^6 \text{kg/cm}^2$; $J = 2400 \text{cm}^4$; $a = 1$ m; $M_0 = 4$ tm; $P = 2$ t; $q = 2$ t/m.

5.86. The bent shown in the figure is loaded by forces $P_1 = 1$ t and $P_2 = 2$ t. The second moment of area of the cross-section is constant over the whole length, and equal to 3000 cm^4. Determine the displacement of end A of the beam in the direction of the force P_1, and the angle of rotation of section B. $E = 2 \times 10^6$ kg/cm^2.

Answer: 4·45 cm; 0·0133.



PROBLEM 5.86

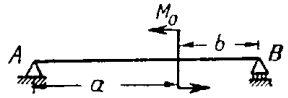
5.87. A beam ABC 5 m long has supports at A and B 3 m apart. When not loaded, end C is on the same level with A and B. A uniformly distributed load with an intensity of 3 t/m acts over the whole length of the beam. What is the value of a load which, when applied to the mid-point of the distance between A and B, will cause point C to return to its initial position?

Answer: 10 t.

5.88. A simply supported beam (see figure) is loaded by a couple M_0. Determine the magnitude of the deflection at the point of application of M_0, and the slope of the end sections A and B.

Answer: $\dfrac{M_0 ab(a-b)}{3EJ(a+b)}$; $\dfrac{M_0}{EJ}\dfrac{a^2+2ab-2b^2}{6(a+b)}$;

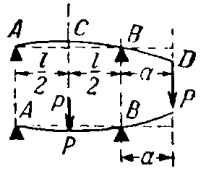
$\dfrac{M_0}{EJ}\dfrac{b^2+2ab-2a^2}{6(a+b)}$.



PROBLEM 5.88

5.89. A beam with an overhang, lying freely on supports A and B, is in one case bent by a force P applied to point D, and in the other case by the same force applied to point C at the mid-span (see figure). Show that the deflection at point C in the first case of loading is equal to the deflection of point D in the second case.

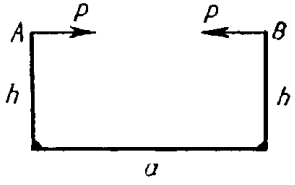
Answer: Both deflections are $\dfrac{Pal^2}{16EJ}$.



PROBLEM 5.89

5.90. Find the reduction in the distance between points A and B of the structure shown in the figure. The cross-section of all portions of the rod is the same. Solve the same problem, applying couples M_0 instead of forces P.

Answer: $\dfrac{Ph^2}{3EJ}(2h+3a)$; $\dfrac{M_0 h}{EJ}(h+a)$.



PROBLEM 5.90

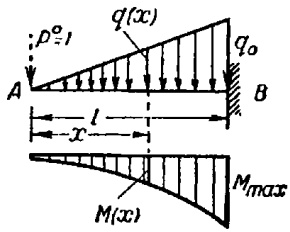
5.91. Determine the value of the deflection of the free end A of cantilever AB loaded by a triangular-shaped load. Determine also the slope at section A.

Solution. We shall apply the method of Maxwell–Mohr. The intensity of the load at any section is $q(x) = q_0(x/l)$; the value of the bending moment is $M(x) = -q_0(x^3/6l)$. We apply a unit force $P^0 = 1$ to point A; $M^0 = -x$. The required deflection is

$$f_A = \frac{1}{EJ}\int_0^l -\frac{q_0 x^3}{6l}(-x)\,dx = \frac{q_0 l^4}{30EJ}.$$

To determine the slope at section A we shall apply to this section a couple $M^0 = 1$; then M^0 has the same value for any section. The required slope is

$$\theta_A = \frac{1}{EJ}\int_0^l -\frac{qx^3}{6l}(-1)\,dx = \frac{q_0 l^3}{24EJ}.$$



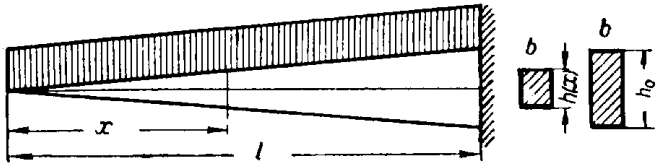
PROBLEM 5.91

5.92. A beam 3 m in span has on the right an overhang of the same length. A concentrated force of 1·5 t is applied to the mid-span. The end of the overhang is prevented from rising by a strap. The right-hand support sinks as long as the strap exerts pressure on the beam. Determine the amount by which the support has sunk. Neglect the weight of the beam. $E = 1 \times 10^5$ kg/cm^2, $J = 7500$ cm^4.

Answer: 1·7 cm.

23. BEAMS OF VARIABLE CROSS-SECTION

5.93. Determine the shape of a cantilever of constant strength loaded by a uniformly distributed load q. The length of the cantilever is l; its cross-section is a rectangle with a constant width b and a variable depth $h(x)$. Find the maximum deflection, and compare it with the deflection (f_0) of a cantilever of uniform cross-section.



PROBLEM 5.93

Solution. The condition for a constant flexural strength is

$$\sigma_{\max} = \frac{M(x)}{W(x)} = \frac{M_0}{W_0} = [\sigma].$$

Substituting the values for the bending moments and the section moduli, we have

$$\frac{qx^2 \times 6}{2 \times b[h(x)]^2} = \frac{ql^2 \times 6}{2 \times bh_0^2}.$$

We establish the equations expressing the variation of the depth of the cross-section, and, that of the second moment of area

$$h(x) = h_0 \frac{x}{h}; \quad J(x) = J_0 \frac{x^3}{l^3}.$$

We write down the equation of flexure of the beam

$$EJ(x) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{qx^2}{2}; \quad EJ_0 \frac{x^3}{l^3} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{qx^2}{2};$$

$$EJ_0 y'' = -\frac{ql^3}{2} \frac{1}{x}.$$

Integrating this equation, we have

$$EJ_0y' = -\frac{ql^3}{2}\log_e x + C.$$

The value of C is determined from the condition that at $x = l$, $y' = 0$:

$$C = \frac{ql^3}{2}\log_e l; \quad \text{then} \quad EJ_0y' = -\frac{ql^3}{2}\log_e x + \frac{ql^3}{2}\log_e l.$$

Integrating again, we have

$$EJ_0y = -\frac{ql^3}{2}x\log_e x - \frac{ql^3}{2}x - \frac{ql^3}{2}x\log_e l + D.$$

From the condition that at $x = l$, $y = 0$ we find D:

$$D = \frac{ql^4}{2}.$$

The final expressions for the deflection, and the magnitude of the maximum deflection are respectively

$$EJ_0y = -\frac{ql^3}{2}(x\log_e x - x - x\log_e l + l)$$

and

$$f_{\max} = -\frac{ql^4}{2EJ_0}, \quad \text{i.e.} \quad f_{\max} = 4f_0.$$

5.94. Two cantilevers of constant strength have an identical cross-section at the fixed end, and are each loaded by an equal force at the free end. The first beam has a variable width, whereas the second has a variable depth. Which beam will be lighter? For which beam is the maximum deflection greater?

Answer: $v_1 : v_2 = f_1 : f_2 = 3 : 4$.

5.95. How will the maximum shear stress vary along the length of each of the two beams in the preceding problem?

Answer: $\tau_1 = \tau_0\frac{l}{x}$; $\quad \tau_2 = \tau_0\sqrt{\frac{l}{x}}$, $\quad$ where $\quad \tau_0 = \frac{3}{2}\frac{P}{a^2}$.

5.96. Determine how the depth of a cantilever with a square cross-section and of constant strength must vary, if it carries a uniformly distributed load q. The allowable stress is $[\sigma]$. Find the deflection of the free end.

Answer: $[a(x)]^3 = \dfrac{3qx^2}{[\sigma]}$; $f = -\dfrac{3ql^4}{8EJ_0}$.

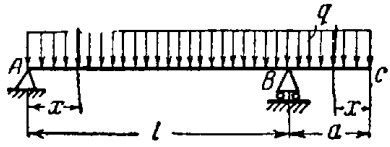
5.97. According to what law must the depth $h(x)$ of the rectangular cross-section of a cantilever vary, so that when loaded by a force at the free end, it would be bent in a circular arc?

Answer: $h(x) = h_0 \sqrt[3]{\dfrac{x}{l}}$.

5.98. Find the law according to which the diameter of beam ABC (see figure) should vary, if the maximum normal stress in any cross-section is to be equal to the allowable stress.

Answer: For section AB: $d^3 = \dfrac{16qx}{\pi l[\sigma]}(l^2 - a^2 - lx)$.

For section BC: $d^3 = \dfrac{16qx^2}{\pi[\sigma]}$.



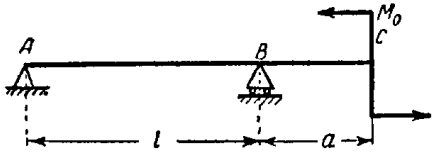
PROBLEM 5.98

5.99. A steel strip 128 cm long, with a width b (= 5 cm), has a variable thickness. At mid-span the thickness t is $t_0\sqrt{2x/l}$, where x is the distance of the section from the end of the beam. The strip is simply supported and loaded at mid-span by a force P. Find the value of the load, P, which causes a deflection of 2·5 cm at mid-span. The modulus of elasticity E is 2×10^6 kg/cm^2.

Answer: 31.7 kg.

5.100. Find the shape of the beam of constant strength shown in the figure. The cross-section of the beam is a rectangle with a depth h and a variable width $b(x)$. Find the deflection at section C.

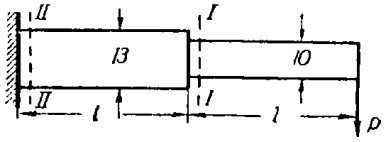
Answer: $\dfrac{M_0 a}{2EJ}(l + a)$.



PROBLEM 5.100

5.101. A cantilever carries a concentrated force P at the free end. One half of the bar, nearest to the constraint has a cross-section 13 cm in diameter; the other half is 10 cm in diameter. Where is the critical section of the bar? (Take no account of stress concentrations in the calculation.)

Answer: The stresses in section II–II are 0·91 times the stresses in section I–I.



PROBLEM 5.101

5.102. Find the value of the end deflection of the beam in Problem 5.101, taking P as 100 kg, l as 40 cm, and E as 10^5 kg/cm^2.

Solution. We shall apply Castigliano's theorem. The bending moments in both sections due to the force P, and their derivatives are respectively

$$M_1 = Px; \quad \frac{\partial M_1}{\partial P} = x; \quad M_2 = Px; \quad \frac{\partial M_2}{\partial P} = x.$$

The second moments of area are

$$J_1 = 490 \text{ cm}^4; \quad J_2 = 1400 \text{ cm}^4; \quad J_2 : J_1 = 2{\cdot}86.$$

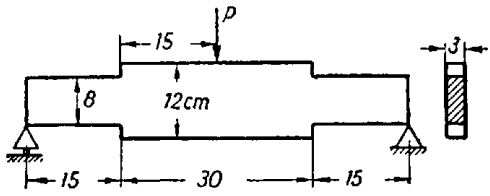
The deflection of the end section of the beam is

$$f = \frac{1}{EJ_1}\int_0^l Px^2\,dx + \frac{1}{EJ_2}\int_0^l Px^2\,dx = \frac{Pl^3}{3EJ_1}\left(1 + \frac{7}{2{\cdot}86}\right)$$

$$= 0{\cdot}15 \text{ cm}.$$

5.103. A beam of variable cross-section (see figure) is loaded by a force P, of 400 kg, at the mid-span. The width of the beam is 3 cm. Determine the value of the maximum normal stress and the maximum shear stress, without taking stress concentrations into account.

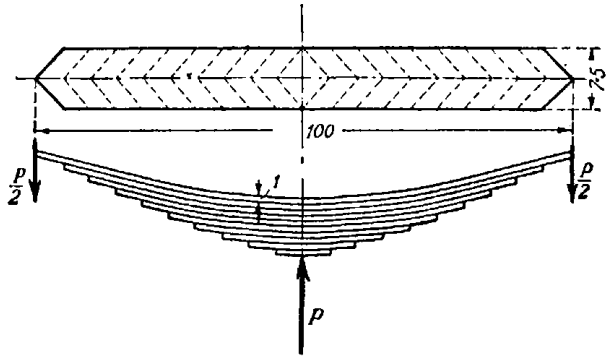
Answer: $\sigma_{max} = 94$ kg/cm^2; $\tau_{max} = 12{\cdot}5$ kg/cm^2.



PROBLEM 5.103

5.104. The spring shown in the figure consists of 10 leaves 7·5 cm wide and 10 mm thick. The span l of the spring is 1 m. The allowable stress is 4000 kg/cm^2. The modulus of elasticity of the spring steel is 2×10^6 kg/cm^2. Determine the load-carrying capacity of the spring, and the value of the deflection at the mid-span.

Answer: $[P] = 2000$ kg; $f = \dfrac{[\sigma]l^2}{4Et} = 5$ cm.



PROBLEM 5.104

5.105. A leaf spring of constant strength, 75 cm long, is made of a strip 7 cm wide and 0·8 cm thick. How many leaves are required, if the spring is to carry a concentrated load of 0·5 t at the middle of the span with an allowable stress of 1500 kg/cm^2? What will

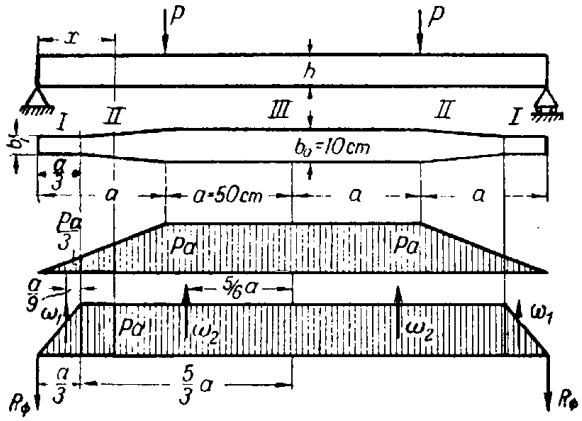
be the maximum deflection if the modulus of elasticity is 2×10^6 kg/cm²?

Answer: 9 leaves; 1·2 cm.

5.106. A pulley, rotating at the mid-span of a shaft of circular cross-section (uniform over the length of the shaft), transmits a force of 1500 kg to the shaft, which is thus subjected to bending. Determine the diameter of the shaft for an allowable stress of 600 kg/cm², without taking the weight of the shaft into account. The span of the shaft is 2·4 m. Determine the deflection (f_1) of the shaft at the section under the pulley. Calculate this deflection (f_2), if the shaft is given a form of constant flexural strength.

Answer: $d = 11{\cdot}5$ cm; $f_1 = 0{\cdot}25$ cm; $f_2 = \frac{3}{80}\frac{Pl^3}{EJ_0} = 0{\cdot}45$ cm.

5.107. A beam (pine) of variable cross-section has the shape and the dimensions shown in the figure. The force P is 200 kg. The allowable stresses $[\sigma]$ and $[\tau]$ are 110 kg/cm² and 12 kg/cm² respectively. The modulus of elasticity E of pine is 10^5 kg/cm². Find the depth h, and the minimum width b of the beam. Determine the deflection at the mid-span of the beam.



PROBLEM 5.107

Solution. The depth of the beam, $h = 7{\cdot}5$ cm, is found from the condition of limiting strength based on normal stress. The

minimum width of the beam, $b_1 = 3{\cdot}33$ cm, is found from the condition of limiting strength based on shear stress.

To determine the deflection, we shall apply the semi-graphical method. We construct the bending moment diagram. The beam consists of three parts: I—where the width is constant, and is one-third of the maximum width b_0, II—where the width varies from b_1 to b_0, and III—where the width is constant. The second moments of area over these sections are respectively $J_1 = {}^1/_3 J_0$, $J_2 = J(x) = J_0(x/a)$, $J_3 = J_0$.

To determine the deflections the formula $f = M_\Phi/EJ$ is used.

We shall replace the $M(x)$ diagram with a diagram which at all sections has been referred to the same second moment of area J_0. For this, ordinates of the $M(x)$ diagram over sections I will be multiplied by factor 3, those over sections II will be multiplied by ratio a/x, whereas those over section III will be left unchanged. The diagram thus corrected is shown at the bottom of the figure.

We determine the fictitious reaction R_Φ, and the areas ω_1 and ω_2:

$$R_\Phi = \frac{11}{16} Pa^2; \quad \omega_1 = \frac{Pa^2}{6}; \quad \omega_2 = \frac{5}{3} Pa^2.$$

The value of the fictitious bending moment at mid-span is

$$M_\Phi = -\left(\frac{11}{6} Pa^2 \times 2a\right) + \frac{1}{6} Pa^2 \left(\frac{5}{3} a + \frac{1}{9} a\right) +$$

$$+ \left(\frac{5}{3} Pa^2 \times \frac{5}{6} a\right) = -\frac{107}{54} Pa^3.$$

The maximum deflection is

$$f_{\max} = -\frac{107}{54} \frac{Pa^3}{EJ_0} = -1{\cdot}41 \text{ cm}.$$

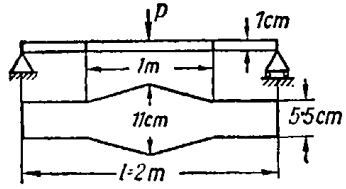
If the width of the beam changes by linear low, then

$$f_{\max} = -\frac{2Pa^3}{EJ_0} = -1{\cdot}42 \text{ cm}.$$

Answer: 7·5 cm; 3·33 cm; 1·41 cm.

5.108. Calculate the maximum deflection of a steel strip subjected to a force P (= 10 kg), and find the maximum stress. The dimensions are given in the figure.

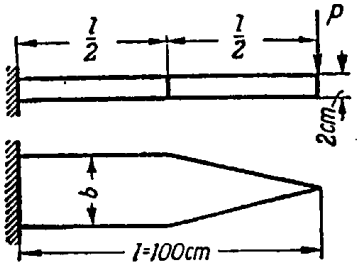
Answer: $\frac{11}{384}\frac{Pl^3}{EJ_0} = 1{\cdot}25$ cm; 273 kg/cm².



PROBLEM 5.108

5.109. Determine the required width of a steel beam for an allowable stress $[\sigma]$ of 1500 kg/cm². Calculate the value of the maximum deflection. The dimensions and the outline of the beam are shown in the figure. The force P is 40 kg.

Answer: $b = 4$ cm; $f = \frac{17}{48}\frac{Pl^3}{EJ_0}$ cm.



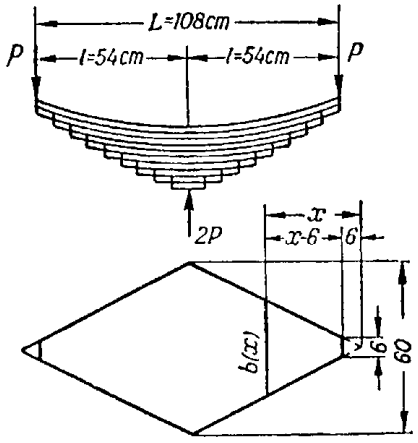
PROBLEM 5.109

5.110. Determine the required width of a timber beam of constant strength, clamped with one end in a wall and carrying a load of 300 kg at the free end, 1·2 m from the wall. The depth of the beam over the whole length is 10 cm. The allowable stress is 100 kg/cm² in flexure, and 12 kg/cm² in shear. Find the deflection of the free end of the beam, taking E as 10^5 kg/cm².

Answer: $b_{\min} = 4$ cm, over length $a = 22$ cm from the free end; $b_0 = 22$ cm at the constraint; $f = P/6EJ_0\,(11a^3 + 3l^3 - 2la^2) = 1{\cdot}41$ cm. If $a = 0$ and $b_{\min} = 0$, then $f = Pl^3/2EJ_0 = 1{\cdot}42$ cm.

5.111. A leaf spring consists of nine leaves forming a plate of uniform thickness having a constant flexural strength, and a tenth

leaf having the same dimensions as the longest one of the other nine. The width of each leaf is 6 cm; the thickness is 1 cm. The span L of the spring is 108 cm, the modulus E is 2×10^6 kg/cm^2. Find the deflection of the spring due to a load, P, of 1·2 t.



PROBLEM 5.111

Solution. We represent (at the bottom of the figure) the spring as a plate having a variable width. The width of the plate at mid-span is $10 \times 6 = 60$ cm. Its width at the ends is equal to the width of the ninth leaf, i.e. 6 cm. The distance x to any section will be taken from the points where the directions of the side faces intersect, i.e. 6 cm from the ends of the spring. We shall carry out the following calculations:

$$b(x) = b_0 \frac{x}{l + x} = 60 \times \frac{x}{60} = x;$$

$$J(x) = J_0 \frac{x}{l + x}; \quad M(x) = P(x - 6); \quad \frac{\partial M(x)}{\partial P} = x - 6;$$

$$f_P = \int_6^{60} \frac{M(x)}{EJ(x)} \frac{\partial M(x)}{\partial P} \, dx$$

$$= \frac{P(l + 6)}{EJ_0} \left\{ \left[\frac{x^2}{2} \right]_6^{60} - [12x]_6^{60} + 36 [\log_e(x)]_6^{60} \right\} = 8{\cdot}6 \text{ cm}.$$

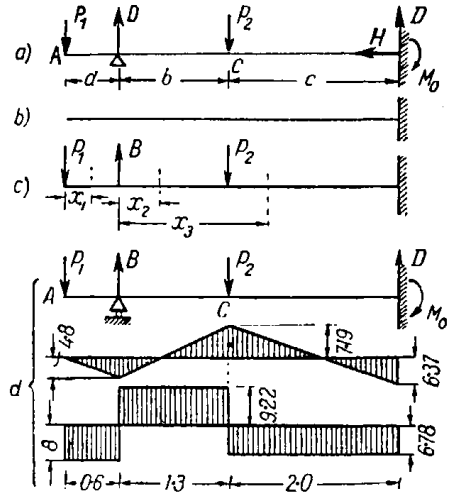
The deflection of the spring of nine leaves (without the top one) for the same conditions is expressed by the formula

$$f_P = \frac{Pl^3}{2EJ_0} = \frac{1200 \times 54 \times 54 \times 54 \times 12}{2 \times 2 \times 10^6 \times 9 \times 6 \times 1^3} = 10{\cdot}5 \text{ cm}.$$

Answer: 8·6 cm.

24. CALCULATION OF STATICALLY INDETERMINATE SYSTEMS

5.112. Draw bending moment and shear force diagrams for the steel beam shown in the figure. Choose an I-section, if the allowable normal stress is 1000 kg/cm². $P_1 = 8$ t; $P_2 = 16$ t; $a = 60$ cm; $b = 130$ cm; $c = 2$ m; $l = b + c = 3{\cdot}3$ m.



Problem 5.112

Solution. The possible reactions at the supports are B, D, H, and M_0. The problem is statically indeterminate to the first degree. We set up the equations of equilibrium

$$H = 0; \qquad B + D = P_1 + P_2;$$

$$-P_1(a + b + c) + B(b + c) - P_2 c + M_0 = 0.$$

Reaction B is taken as the unknown redundant. The basic diagram of the beam is shown in figure (b). We load this beam by the given forces P_1 and P_2 and the additional unknown force B (figure (c)). To make the beam in figure (c) equivalent to the given actual beam (figure (a)), we specify (for the beam in figure (c))

$$f_B = 0.$$

The condition $f_B = 0$ is fulfilled by means of Castigliano's theorem

$$f_B = \frac{\partial U}{\partial B}$$

$$= \frac{1}{EJ}\left\{\int_0^a M_1 \frac{\partial M_1}{\partial B}\,dx + \int_0^b M_2 \frac{\partial M_2}{\partial B}\,dx + \int_b^{b+c} M_3 \frac{\partial M_3}{\partial B}\,dx\right\} = 0,$$

where M_1, M_2 and M_3 are the bending moments in the three parts of the beam. The rigidity EJ is taken as constant over the length of the beam.

We calculate the quantities under the integral signs

$$M_1 = -P_1 x_1; \qquad \frac{\partial M_1}{\partial B} = 0;$$

$$M_2 = -P_1(x_2 + a) + Bx_2; \qquad \frac{\partial M_2}{\partial B} = x_2;$$

$$M_3 = -P_1(x_3 + a) + Bx_3 - P_2(x_3 - b); \qquad \frac{\partial M_3}{\partial B} = x_3.$$

We substitute these values of M and $\partial M/\partial B$ in the equation $f_B = 0$

$$\int_0^b (-P_1 x^2 - P_1 a x + Bx^2)\,dx + \int_0^{b+c} (-P_1 x^2 - P_1 a x + Bx^2 - P_2 x^2 + P_2 b x)\,dx = 0;$$

$$\int_0^{b+c} (-P_1 x^2 - P_1 a x + Bx^2)\,dx + \int_0^{b+c} (-P_2 x^2 + P_2 b x)\,dx = 0.$$

After integrating and solving the equation for B, we have

$$B = P_1\left(1 + \frac{3}{2}\frac{a}{b+c}\right) + P_2\left(1 - \frac{3}{2}\frac{b}{b+c} + \frac{b^3}{2(b+c)^3}\right).$$

Inserting the values of a, b, and c, and taking into account that $P_2 = 2P_1$, we find that

$$B = 2{\cdot}15P_1 = 17{\cdot}22\text{ t}.$$

From the conditions of equilibrium

$$D = P_1 + P_2 - B = 8 + 16 - 17{\cdot}22 = 6{\cdot}78\text{ t};$$

$$M_0 = P_1(a+b+c) + P_2c - B(b+c)$$
$$= (8 \times 3{\cdot}9) + (16 \times 2) - (17{\cdot}22 \times 3{\cdot}3) = 6{\cdot}37\text{ tm}.$$

Knowing all reactions at the supports, we construct the M and Q diagrams (figure (d)). $M_{\max} = +7{\cdot}19$ tm, and the required section modulus

$$W = \frac{719{,}000}{1000} = 719\text{ cm}^3.$$

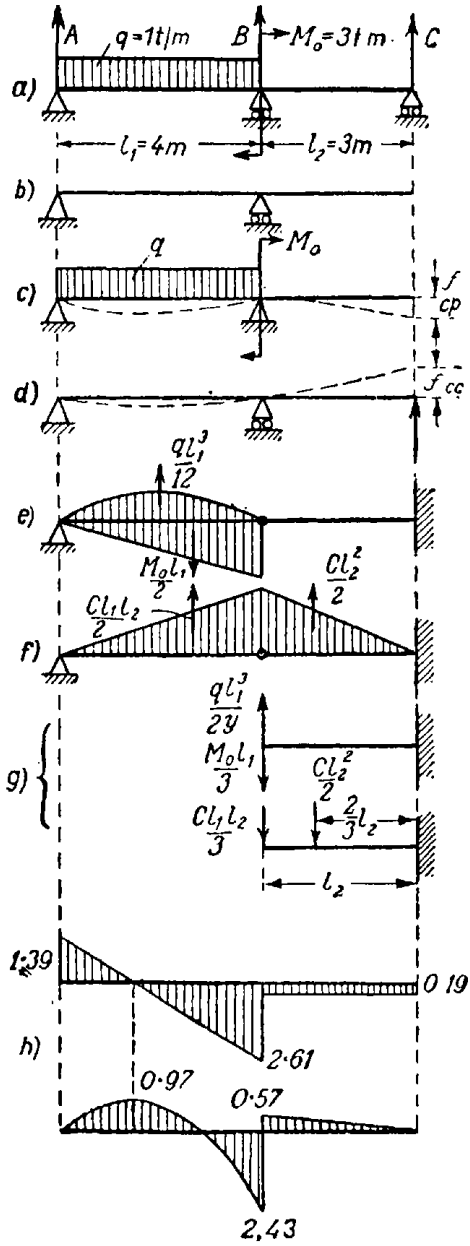
A No. 33a I-section has $W = 721$ cm^3.

5.113. Determine the reactions at the supports of the beams shown in the figure (diagram (a)). Construct bending moment and shear force diagrams.

Solution. To overcome the static indeterminancy we shall employ the method of compatibility of deflections. We choose the support C as the redundant constraint. The diagram of the statically determinate beam is shown in figure (b). The beam is now loaded by the given load (diagram (a)) and the redundant unknown force C (diagram (c)). The deflection of the beam at point C, due to the action of the given load, is denoted by f_{cp}. The deflection due to the force C is denoted by f_{cc}. Thus at point C of the support of the given beam $f_c = f_{cp} + f_{cc} = 0$.

To calculate the deflections it is convenient to employ the semi-graphical method. We construct the bending moment diagrams

for the loads q and M (diagram (e)), and for the force C (diagram (f)). We take the zero line as the axis of the fictitious beams,



PROBLEM 5.113

and indicate the constraints of these fictitious beams. We then calculate the areas of these diagrams, and determine the thrust being transmitted from the left-hand fictitious beam to the right-hand beam via hinge B. Further we consider only the right-hand beam (diagram (g)). We write down the sum, equal to zero, of the bending moments at section C

$$\left(\frac{ql_1^3}{24} \times l_2\right) - \left(\frac{M_0 l_1}{3} \times l_2\right) + \left(\frac{Cl_1 l_2}{3} \times l_2\right) + \left(\frac{Cl_2^2}{2} \times \frac{2}{3} l_2\right) = 0.$$

From this equation we have

$$C = \frac{l_1}{l_2} \times \frac{M_0 - \frac{ql_1^2}{8}}{l_1 + l_2} = 0{\cdot}19 \text{ t}.$$

From the equations of equilibrium for the beam in diagram (a) we have

$$A = 1{\cdot}39 \text{ t}; \qquad B = 2{\cdot}42 \text{ t}.$$

We now plot by the usual methods the M and Q diagrams which are shown in figure (h).

5.114. The beam ABC is built in at end B and supported on a movable support at end A. A force P of 4 t is applied to the mid-span of the beam. Plot bending moment and shear force diagrams.

Solution. We shall employ Mohr's method to overcome the static indeterminancy. We choose support A as the reduntant constraint. In the diagram used for the calculation the statically determinate beam is loaded by the given force P and the redundant unknown force A (diagram (b)). We load an identical beam by a force $P^0 = 1$. It is applied to point A and has the same direction as the force A (diagram (c)).

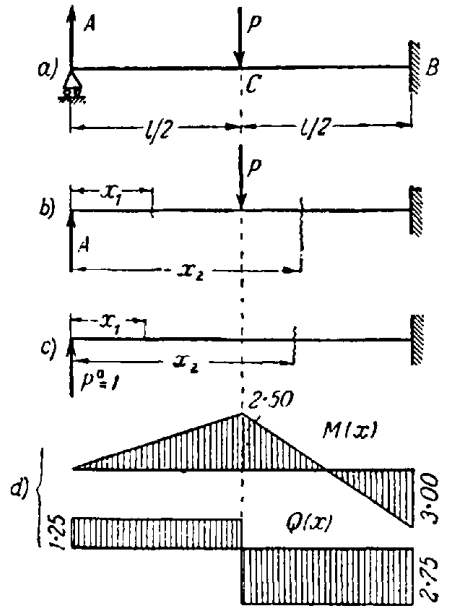
We calculate the values of the bending moments $M(x)$ and $M^0(x)$

$$\begin{aligned}
&\text{Part I:} && M(x) = A x_1; \\
& && M^0(x) = x_1; \\
&\text{Part II:} && M(x) = A x_2 - P(x_2 - 0{\cdot}5\, l); \\
& && M^0(x) = x_2.
\end{aligned}$$

We calculate Mohr's integrals and equate their sum to zero

$$\int_0^{0.5l} A x_1^2 \, dx + \int_{0.5l}^{l} [A x_2^2 - P(x_2 - 0.5\, l) x_2] \, dx = 0.$$

From this equation we find that $A = 5/16P$. Knowing A, it is possible to plot by the usual methods the M and Q diagrams shown in diagram (d).

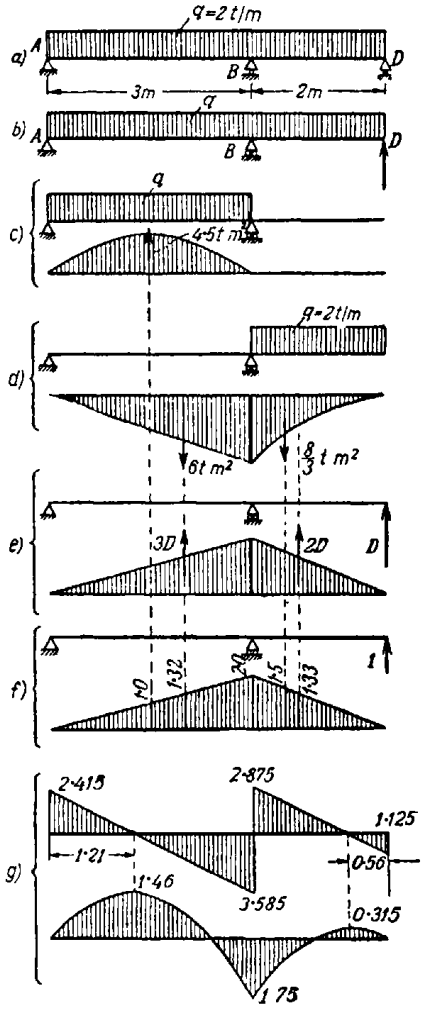


PROBLEM 5.114

5.115. Construct the bending moment and shear force diagrams for the beam ABD shown in the figure.

Solution. To overcome the static indeterminancy we shall employ Vereshchagin's method. The right-hand support is chosen as the redundant constraint. The beam diagram adopted, together with the complete loading (due to the given forces, and the redundant unknown force D) is shown in figure (b). The condition of compatibility of deformations is $f_D = 0$. When using Vereshchagin's formula this condition is re-written as $\sum \omega \times M_D^0 = 0$. To evaluate this sum we represent the beam with each of these loads separately, and plot bending moment diagrams for them. We plot

a bending moment diagram for the load q on span AB (diagram (c)), the load q on overhang BD (diagram (d)), the redundant unknown D (diagram (e)), and the load $P^0 = 1$ applied to point D in the direction of force D (diagram (f)). In the first three diagrams we show the values of the diagram areas, and the positions of the centroids of these areas. In the fourth diagram (f) we show the ordinates of M_C^0 corresponding to the centroids of the areas of the preceding diagrams.



PROBLEM 5.115

Thus the condition of compatibility is

$$(4{\cdot}50 \times 1) - \left(6 \times \frac{4}{3}\right) - \left(\frac{8}{3} \times 1{\cdot}5\right) + \left(5D \times \frac{4}{3}\right) = 0.$$

Hence $D = 1{\cdot}125$ t.

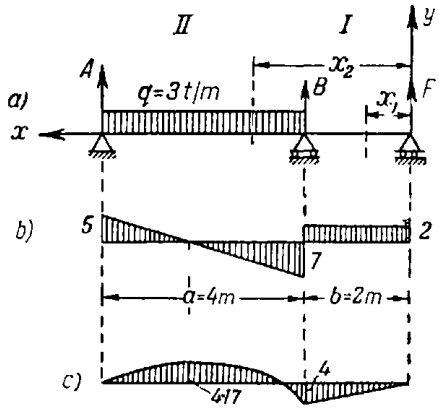
From the equations of equilibrium for the beam (a) we find the values of the remaining reactions.

$$A = 2{\cdot}415 \text{ t};$$

$$B = 6{\cdot}460 \text{ t}.$$

The M and Q diagrams, constructed by the usual methods, are shown in diagram (g).

5.116. Determine the reactions at the supports of the beam shown in the figure. Plot bending moment and shear force diagrams.



PROBLEM 5.116

Solution. We shall employ the method of initial parameters. The general equation of deflections is

$$EJy = EJy_0 + EJ\theta_0 \times \frac{x}{1!} + \sum \frac{M(x - l_0)^2}{2!}$$

$$+ \sum \frac{P(x - l_0)^3}{3!} + \sum \frac{q(x - l_0)^4}{4!}.$$

We apply it to the supports A and B

$$EJ\theta_A \times b + \frac{Fb^3}{6} = 0; \tag{1}$$

$$EJ\theta_A(b + a) + \frac{F(b + a)^3}{6} + B\frac{a^3}{6} - q\frac{a^4}{24} = 0. \tag{2}$$

We add the equation of equilibrium

$$F(a + b) + Ba - \frac{qa^2}{2} = 0. \tag{3}$$

Having determined $EJ\theta_A$ from (1), we substitute it in (2)

$$-F\frac{b^2}{6}(b + a) + \frac{F(b + a)^3}{6} + B\frac{a^3}{6} - q\frac{a^4}{24} = 0. \tag{4}$$

We multiply (3) by $a^2/6$ and substract it from (4)

$$-F\frac{b^2}{6}(b + a) + \frac{F(b + a)^3}{6} - F\frac{a^2}{6}(b + a) = \frac{qa^4}{24} - \frac{qa^4}{12}.$$

Solving this equation, we have

$$F = -\frac{qa^3}{8b(a + b)} = -2\text{ t}.$$

It follows from (3) that $B = 9$ t. We determine the point in section II, at which the bending moment is the maximum, and then find the bending moment itself

$$-F - B - q(x_0 - b) = 0;$$

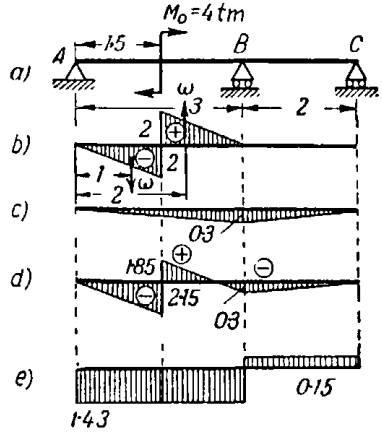
$$2 - 9 - 3x_0 - 6 = 0;$$

$$x_0 = 4{\cdot}333\text{ m};$$

$$M_{max} = -\left(2 \times \frac{13}{3}\right) + \left(9 \times \frac{7}{3}\right) - \left(3 \times \frac{49}{9 \times 2}\right) = \frac{25}{6}$$

$$= 4{\cdot}17\text{ t m}.$$

5.117. A two-span continuous beam is loaded as shown in figure (a). Determine the reactions at the supports, and plot bending moment and shear force diagrams. Determine the cross-section for the beam, assuming it to be rectangular with the ratio $h : b = 2$. Take the allowable normal stress in flexure as 110 kg/cm².



PROBLEM 5.117

Solution. We shall apply the theorem of three moments to overcome the static indeterminancy. We plot the bending moment diagram for the left-hand span as for a simple beam (figure (b)). The distances from the left-hand support to each resultant of the fictitious load are shown in the figure.

We apply the formula of three moments for the left-hand support B

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = -6\left(-\omega \times \frac{1}{3} + \omega \times \frac{2}{3}\right).$$

The moments at the end supports are zero, consequently

$$2M_B(l_1 + l_2) = -\frac{6}{3}\omega;$$

$$M_B = -0{\cdot}3 \text{ tm}.$$

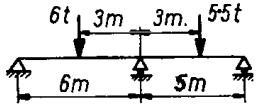
We plot the diagram of the support moments (figure (c)), and add it to the diagram of the left-hand simple beam (figure (b)). The resultant diagram is shown in figure (d).

We determine the reactions at the supports, and construct the shear force diagram shown in figure (e).

The required section modulus W is 1950 cm^3, whence the required depth of the cross-section h is 28·8 cm $\approx$ 29 cm.

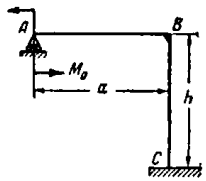
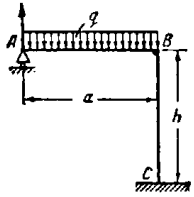
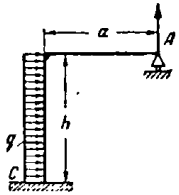
5.118. Determine the reactions at the supports, and find the maximum and minimum bending moments for the beam shown diagrammatically in the figure. Choose an I-section if the allowable stress is 1600 kg/cm^2.

Answer: The reactions are 2·04 t, 7·31 t, 2·15 t. The bending moments are 6·12 tm, —5·76 tm. The section is I No. 24a.



PROBLEM 5.118

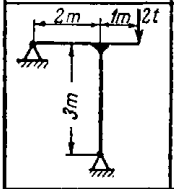
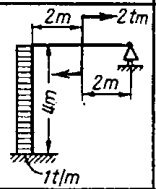
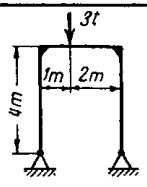
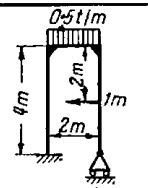
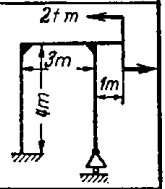
5.119. Plot bending moment and shear force diagrams for frames loaded as shown schematically in the following table.

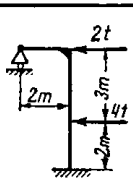
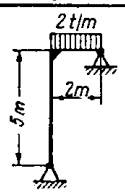
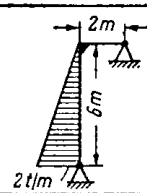
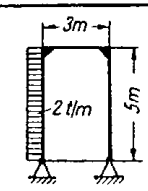
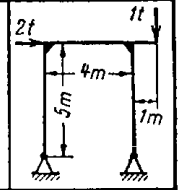
Diagram	Numerical data	*Answer*
	$M_0 = 4$ tm $a = 2$ m $h = 4$ m	$A = \frac{3}{2}\frac{M_0}{a}\frac{a+2h}{a+3h} = 2{\cdot}14$ t $M_z = 0{\cdot}28$ tm
	$q = 2$ t/m $a = 2$ m $h = 6$ m	$A = \frac{3}{8}qa\frac{a+4h}{a+3h} = 1{\cdot}95$ t $M_{\max} = 0{\cdot}95$ tm
	$q = 2$ t/m $a = 2$m $h = 5$m	$A = \frac{qh^3}{2a(a+3h)} = 3{\cdot}67$ t $M_C = -17{\cdot}67$ tm

PROBLEM 5.119

5.120. Determine the reactions at the supports of the frames shown in the table below applying any method to overcome the static indeterminancy. Construct bending moment, shear force and axial force diagrams.

The answers are given underneath the frame diagrams for the maximum (in absolute value) bending moments and shear forces.

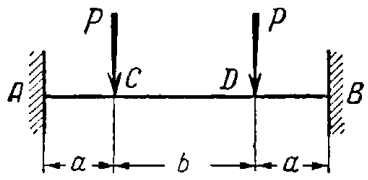
a	b	c	d	e
				
2 t m	5·94 t m	1·47 t m	2·39 t m	2t m
2t	4t	2t	1t	0·734t

f	g	h	i	j
				
12·18 t m	0·86 t m	3·40 t m	14·15 t m	5·73 t m
6·00t	2·14t	3·48t	8·33t	2·75t

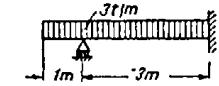
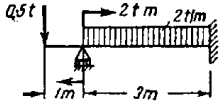
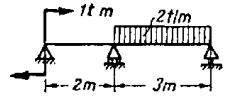
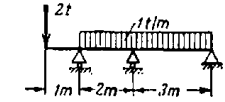
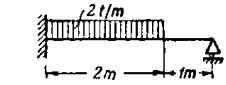
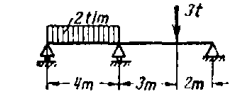
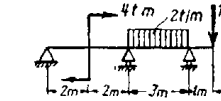
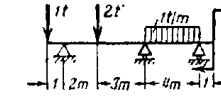
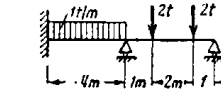
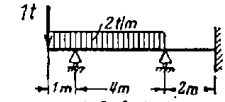
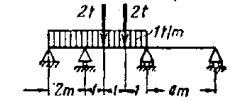
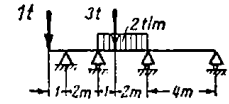
PROBLEM 5.120

5.121. Plot M and Q diagrams for statically indeterminate beams loaded as shown in the table on page 291. Apply any method to overcome the static indeterminancy.

5.122. Choose the cross-section for an I-beam clamped at both ends, and loaded as indicated in the figure. Take $a = 1{\cdot}5$ m, $b = 3$ m, $P = 6$ t. The allowable stress is 1600 kg/cm^2.



PROBLEM 5.122

	Beam Diagram	*Answer*	
		$M^{\max}_{\min}$ tm	$Q^{\max}_{\min}$ t
a		+1·35 −2·62	+4·13 −4·87
b		+1·31 −3·00	+1·50 −4·50
c		+1·54 −1·55	+3·52 −2·48
d		+0·90 −2·00	+1·76 −2·00
e		+0·88 −1·78	+3·26 −0·74
f		+2·57 −3·18	+3·21 −4·79
g		+1·48 −2·52	+3·01 −2·99
h		+1·27 −1·32	+2·08 −1·92
i		+1·54 −1·86	+2·46 −2·20
j		+1·90 −2·18	+3·95 −4·05
k		+1·78 −1·79	+3·77 −3·23
l		+2·42 −1·80	+5·22 −3·78

PROBLEM 5.121

Instruction. Make use of the symmetry of the beam and the loads to overcome the static indeterminancy. Apply the method of compatibility of angular deformations.

Answer: $M_A = -P \times a \times \dfrac{a+b}{2a+b} = -6{\cdot}75\ \text{tm};$

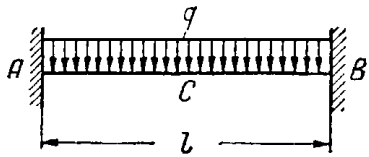
$$M_C = P \times \frac{a^2}{2a+b} = 2{\cdot}25\ \text{tm};$$

I No. 27a.

5.123. Following the instructions given in Problem 5.122 choose the cross-section for an I-beam loaded by a uniformly distributed load $q = 1{\cdot}2$ t/m over a span $l = 4$ m.

Answer: $M_A = -\dfrac{ql^2}{12} = -1{\cdot}6\ \text{tm};\quad M_C = \dfrac{q\,l^2}{24} = 0{\cdot}8\ \text{tm};$

I No. 16.

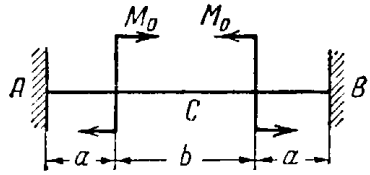


PROBLEM 5.123

5.124. Following the instructions given in Problem 5.122, choose the cross-section for an I-beam loaded by two couples $M_0 = 3$t m, if $a = 2$ m, $b = 4$ m.

Answer: $M_A = -M_0 \dfrac{b}{2a+b} = -1{\cdot}5\ \text{tm};$

$$M_C = M_0 \frac{2a}{2a+b} = 1{\cdot}5\ \text{tm};\ \text{I No. 14.}$$

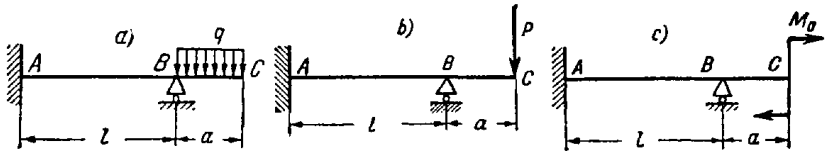


PROBLEM 5.124

5.125. Applying any method, overcome the static indeterminancy, determine the reactions at the supports, and plot the M and Q diagrams.

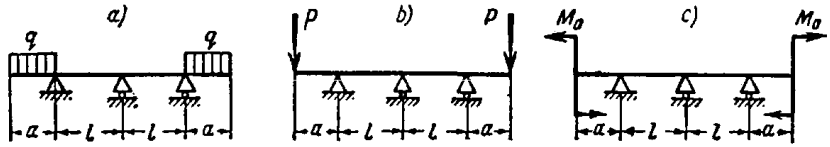
Answer: (a) $B = qa\left[1 + \left(\frac{3}{4} \times \frac{a}{l}\right)\right]$;

(b) $B = P\left[1 + \left(\frac{3}{2} \times \frac{a}{l}\right)\right]$; (c) $B = \frac{3}{2}\frac{M_0}{l}$.



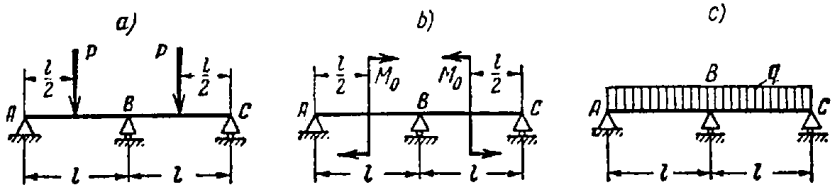
PROBLEM 5.125

5.126. A continuous two-span beam of constant cross-section, with overhangs, is loaded as indicated in the figure. Using the results of the preceding problem, plot the bending moment and shear force diagrams. Check the answer by applying the theorem of three moments.



PROBLEM 5.126

5.127. A two-span continuous beam is loaded as indicated in the figure. Determine the value of the reactions of the middle support by the method of compatibility of linear deformations. Construct the M and Q diagrams.



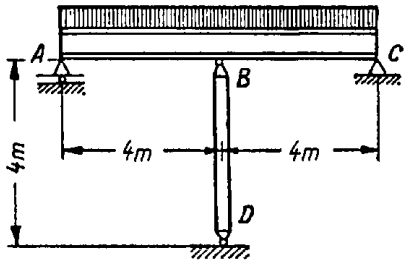
PROBLEM 5.127

Answer: (a) $B = 1{\cdot}375P$, $M_{\max} = -0{\cdot}1875Pl$;

(b) $B = 2{\cdot}25M_0/l$, $M_{\max} = -0{\cdot}5625M_0$;

(c) $B = 1{\cdot}25ql$, $M_{\max} = -0{\cdot}125ql^2$.

5.128. A steel No. 36c I-beam is supported at points A and C on rigid supports, while at point B it is supported on a cast iron hollow column BD. The length of the beam is 8 m, and the height of the column is 4 m; the external diameter of the column is 20 cm, the internal diameter is 15 cm. The beam carries a uniformly distributed load $q = 2$ t/m. Determine the force in column BD. What would be the reaction of the middle support, if it is absolutely rigid?



PROBLEM 5.128

Solution. The equation of compatibility of deformations for the given case is written as follows:

$$\frac{5}{384}\frac{qL^4}{E_sJ} - \frac{BL^3}{48E_sJ} = \frac{BH}{E_{c.i.}F}.$$

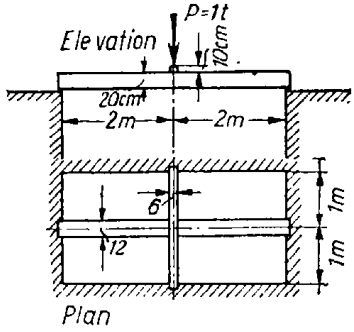
Hence B may be calculated:

$$B = \frac{\frac{5}{8}qL}{1 + \frac{E_s}{E_{c.i.}}\frac{48HJ}{FL^3}} = \frac{\frac{5}{8}qL}{1{\cdot}0091} = 9910 \text{ kg}.$$

With a rigid support at the middle of the span $B = 5/8qL = 10{,}000$ kg.

5.129. A rectangular space is bridged over by two wooden beams, loaded at the point of intersection by a vertical force. The dimensions of the beams are shown in the figure. Determine the maximum normal stress in each of the beams. The beams are simply supported.

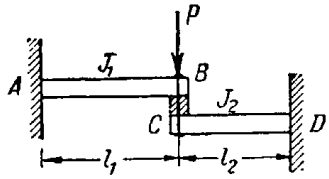
Answer: 167 kg/cm² (in the short beam) and 83 kg/cm².



PROBLEM 5.129

5.130. A force P is applied to the point where two beams AB and CD join each other. Find how the force P is distributed between the beams if the ratio of their spans and rigidities are respectively $l_1 : l_2 = 3 : 2$ and $EJ_1 : EJ_2 = 4 : 5$.

Answer: $P_1 = 0{\cdot}19P$; $P_2 = 0{\cdot}81P$.



PROBLEM 5.130

5.131. Two beams AB and CD of rectangular cross-section have the same width and length, but the depth of beam AB is half the depth of beam CD. The beam CD is simply supported. Three rigid supports are placed on the beam: at each end and at mid-span. The beam AB rests on these supports, and is loaded by a concen-

trated force P at the mid-span of beam CD. Determine the thrust on the supports of beam AB, neglecting its weight.

Answer: $\frac{1}{18}P$; $\frac{8}{9}P$; $\frac{1}{18}P$.

5.132. A gap is bridged by three parallel beams of rectangular cross-section. The width of all beams is the same. The depth of the middle beam is twice the depth of the end beams. Initially the top faces of the beams are in the same horizontal plane. They are then bridged at the middle of the span by an absolutely rigid transverse member, transmitting a concentrated force P to these beams. What proportion of the load is carried by each beam?

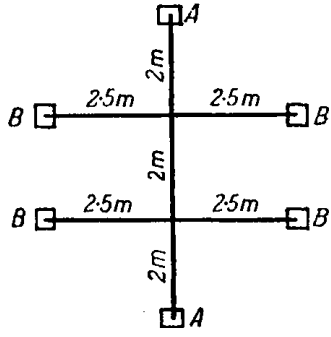
Answer: 1:8:1.

5.133. A beam 5 m long is built into a wall at one end. Its free end is supported at the mid-span of another beam which is simply supported, and has the same length. A force $Q = 6$ t is applied to the first beam at a distance of 4 m from the constraint. Determine the reactions of the simply supported beam.

Answer: 1·99 t.

5.134. In the figure AA represents in plan a simply supported beam, and BB represent two simply supported beams, resting also on beam AA. At the points where the axes of the beams intersect two forces of 400 kg each are applied. All three beams are of identical cross-section. Determine the reactions at the supports A and B.

Answer: $A = 112{\cdot}4$ kg; $B = 143{\cdot}8$ kg.



PROBLEM 5.134

5.135. A continuous beam ABC, with two spans of length l each, is loaded by a uniformly distributed load of intensity q. The reaction of the middle support B is ql. Determine the vertical distance between support B and the straight line connecting supports A and C.

Solution. We denote the value of the reaction of the middle support by B. The required distance then represents the possible deflection of section B. We employ the method of compatibility of deflections. Removing the middle support we can calculate the deflection at point B due to the load q, by using the formula

$$f_{Bq} = \frac{5q(2l)^4}{384EJ} = \frac{5ql^4}{24EJ}.$$

The deflection at the same section, due to the action of the force B, will be calculated from the well known formula

$$f_{BB} = \frac{B(2l)^3}{48EJ} = \frac{ql(2l)^3}{48EJ} = \frac{ql^4}{6EJ}.$$

The possible deflection of section B is

$$f_B = f_{Bq} - f_{BB} = \left(\frac{5}{24} - \frac{1}{6}\right)\frac{ql^4}{EJ} = \frac{ql^4}{24EJ}.$$

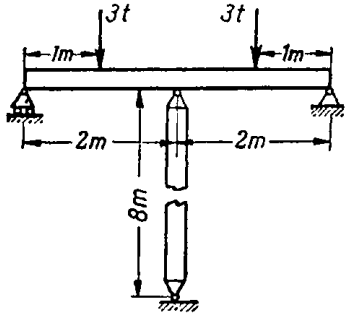
Consequently, the support will be a vertical distance $ql^4/24EJ$ below the end supports.

5.136. A steel beam rests on three supports placed at equal distances apart, and carries a uniformly distributed load with an intensity of 500 kg/m. The reaction of the middle support is 2·5 t. Both spans are 4 m. Find the vertical position of the middle support.

Answer: The supports are located on the same level.

5.137. A simply supported steel No. 20a I-beam is propped at the middle of its span by a wooden post having a square cross-section of 20×20 cm^2 (see figure). Find the maximum normal stress in the beam, and the force in the post.

Answer: The force in the post is 3·85 t. The stress in the beam is 454 kg/cm².



PROBLEM 5.137

5.138. A beam with a length $2l$ of constant cross-section is simply supported at ends A and B, and also at the mid-point C of its length. The supports A and B are rigid, the support C deflects elastically by an amount μC, where C is the load carried by this support. Show that if a load Q is placed on the beam, and is uniformly distributed over the whole length, the load carried by the middle support is

$$\frac{5}{8}\,\frac{Q}{1+\dfrac{6EJ}{l^3}\,\mu}.$$

5.139. A beam of constant cross-section, with a length l and a weight Q, is clamped at one end, and supported simply at the other end on a column, which sinks 1 cm under an axial thrust α. Both supports were located initially on the same level. Due to the weight of the beam the height of the column has been reduced. Calculate the axial thrust in the column.

Answer: $S = \dfrac{3}{8}\,\dfrac{Q}{K}$, where $K = 1 + \dfrac{3EJ}{\alpha l^3}$.

5.140. A beam 4 m long is clamped at one end and loaded by a uniformly distributed force of 400 kg/cm. The beam is propped at a distance of 2·5 m from the clamped end so that the bending moment at the clamped end is zero.

Find the thrust on the support, and plot bending moment and shear force diagrams for the beam.

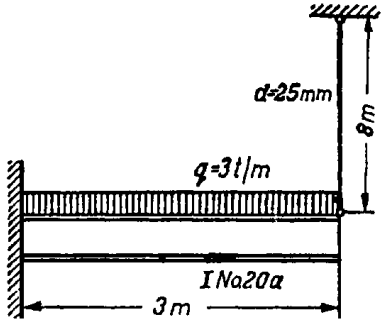
Answer: 1280 kg;

$M_{max} = 0{\cdot}13$ tm; $M_{min} = -0{\cdot}45$ tm;

$Q_{max} = -0{\cdot}68$ t; $Q_{min} = -0{\cdot}6$ t.

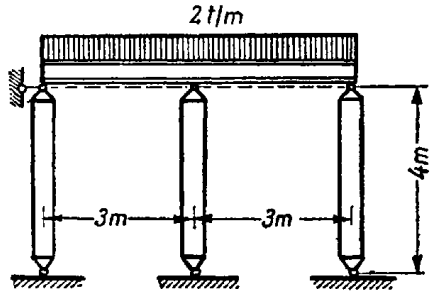
5.141. A balcony is supported by a horizontal I-beam built with one end into the wall; the other end is suspended from a vertical steel rod. The load on the beam is distributed uniformly (see figure). Assuming that the upper constraint of the rod does not move, determine the magnitude of the force in the rod, and the value of the maximum normal stress in the beam. How will the force and the stress change, if the rod is 2·5 mm longer than the designed length.

Answer: The force in the rod: 3240 kg and 3110 kg. The stress in the beam: 738 kg/cm² and 679 kg/cm² in the span; 1595 kg/cm² and 1755 kg/cm² at the fixed end.



PROBLEM 5.141

5.142. A beam with a constant cross-section and a weight Q, rests on three identical elastic stanchions, which are placed under the



PROBLEM 5.142

ends of the beam, and at the mid-point of its length. The force required to shorten the stanchions by 1 cm is α. The length of the beam is l. Show that the load on the middle stanchion is

$$B = \frac{Q}{9} \frac{5l^3\alpha + 192EJ}{l^3\alpha + 72EJ}.$$

5.143. A steel No. 18 I-beam, with a length $l = 6$ m, covers two spans. It rests freely on three wooden posts of circular cross-section of diameter $d = 25$ cm and a height of 4 m. The beam carries a uniformly distributed load with an intensity of 2 t/m.

Determine the maximum normal stress in the beam, and the stresses in the posts, making use of the formula for B in Problem 5.142. $E_s = 2 \times 10^6$ kg/cm^2; $E_t = 10^5$ kg/cm^2.

Answer: $B = 7470$ kg; 1190 kg/cm^2; 4·5 kg/cm^2; 15·2 kg/cm^2; 4·5 kg/cm^2.

5.144. A steel beam, of constant cross-section and a length l, is supported at the ends on rigid foundations, and on steel columns of constant cross-section and length h, placed at points $\frac{1}{3}$ from the ends. When the beam is not loaded all four supports are on the same level.

Show that for a uniformly distributed load over the whole length of the beam to produce equal reactions in all four supports, the cross-sectional areas of the columns must be $486hJ/7l^3$, where J is the second moment of area of the cross-section of the beam.

5.145. A two-span continuous beam, with spans l_1 and l_2, is clamped at one end and freely supported at the other. A load P is placed on the span l_2 at a distance a from the freely supported end. Find the expression for the bending moment at the clamped end and at the middle support.

Answer: $M_{mid} = -2M_3$; $M_3 = \frac{Pa(l_2^2 - a^2)}{l^2(3l_1 + 4l_2)}$.

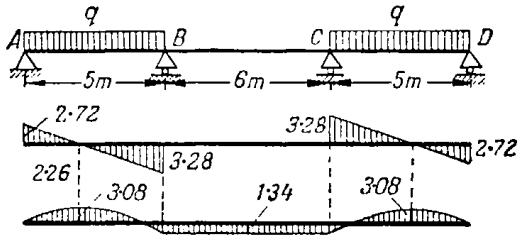
5.146. A continuous beam has two spans, each of length l. Its left-hand end is fixed. Determine the reactions at the supports, if the beam is loaded by a concentrated force P at the middle of the left-hand span.

Answer: $\frac{17}{28}P$; $\frac{25}{56}P$; $-\frac{3}{56}P$.

5.147. A continuous beam has two spans AB and BC, each of length l. Loads $2P$ and P are applied respectively at the middle of span AB and that of span BC. The second moment of area of the cross-section of the portion AB is twice the second moment of area of portion BC. Find the reactions at the supports, and the bending moment at the middle support.

Answer: $\frac{3}{4}P$; $2P$; $\frac{1}{4}P$; $-\frac{Pl}{4}$.

5.148. A No. 20a I-section is used as a three-span continuous beam. It has 5 m long end spans, and a 6 m long middle span. The end spans are loaded by a uniformly distributed load of intensity 1200 kg/m; the middle span is not loaded (see figure). Determine the maximum normal stress in the beam.



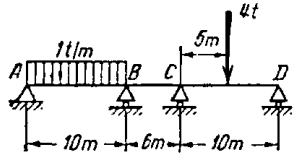
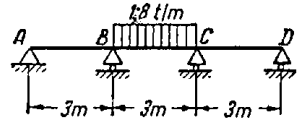
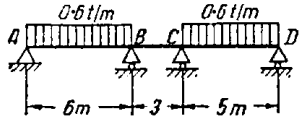
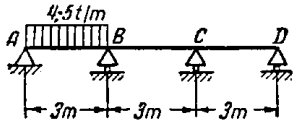
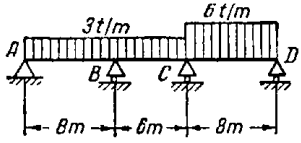
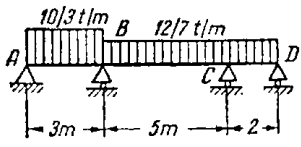
PROBLEM 5.148

Instruction. Due to the symmetry of the beam and the loading $M_B = M_C$; thus it is sufficient to write down only one equation of three moments.

Answer: $M_B = -1{\cdot}34$ tm $= M_C$. The shear force and bending moment diagrams are given in the figure. The maximum stress

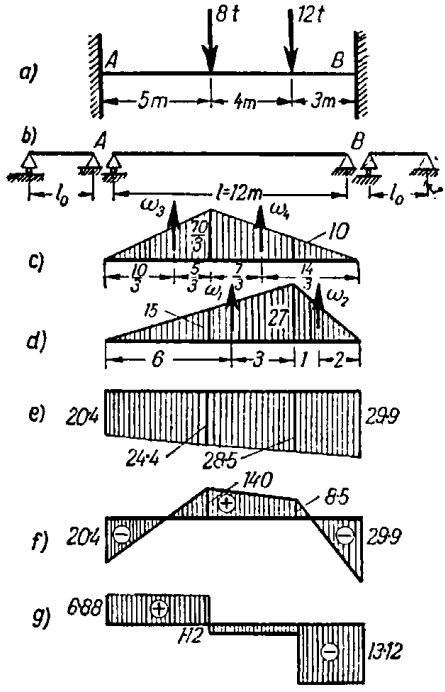
$$\sigma = \frac{M_{max}}{W} = \frac{308{,}000}{237} = 1300 \text{ kg/cm}^2.$$

5.149. A three-span continuous beam of I-section is loaded as indicated in the diagrams tabulated on page 302 Plot bending moment and shear force diagrams. Determine the required I-section, if the allowable stress $[\sigma]$ is 1600 kg/cm^2.

	Diagram of beam and loading	Answer					
		$M\frac{\max}{\min}$ tm	A t	B t	C t	D t	No. of I
a		−7·18 +9·17	4·28	6·36	1·69	1·67	30a
b		−0·81 +1·22	−0·27	2·97	2·97	−0·27	12
c		−1·66 +1·93	1·52	2·35	1·40	1·33	16
d		−2·70 +3·80	5·85	8·78	−1·35	+0·22	20a
e		−30·4 +34·0	10·38	19·72	39·70	20·20	50c
f		−3·92 +2·13	3·69	10·85	7·08	0·38	20b

PROBLEM 5.149

5.150. Draw bending moment and shear force diagrams for a beam having a span of 12 m. It is clamped at both ends and loaded by a force of 8 t at a distance of 5 m from the left-hand support, and by a force of 12 t at a distance of 9 m from the left-hand support (figure (a)).



PROBLEM 5.150

Solution. Replacing the constraints at each end by additional spans we have a calculation diagram in the form of three beams having spans l_0, l and l_0 (figure (b)). We draw a bending moment diagram due to $P_1 = 8$ t and $P_2 = 12$ t (figures (c) and (d)), and calculate the areas ω_1, ω_2, ω_3 and ω_4, which are taken as fictitious loads directed upwards. We indicate the positions of the fictitious resultants by marking the appropriate distances in the figure, and calculate their values

$$\omega_1 = \frac{243}{2}\ \text{t m}^2; \quad a_1 = 6\ \text{m}; \quad b_1 = 6\ \text{m};$$

$$\omega_2 = \frac{81}{2}\,\text{tm}^2; \qquad a_2 = 10\,\text{m}; \quad b_2 = 2\,\text{m};$$

$$\omega_3 = \frac{175}{3}\,\text{tm}^2; \quad a_3 = \frac{10}{3}\,\text{m}; \quad b_3 = \frac{26}{3}\,\text{m};$$

$$\omega_4 = \frac{245}{3}\,\text{tm}^2; \quad a_4 = \frac{22}{3}\,\text{m}; \quad b_4 = \frac{14}{3}\,\text{m}.$$

We apply the formula of three moments twice: for supports A and B

$$M_1 l_0 + 2M_A(l_0 + l) + M_B l = -6\left[\omega_1 \frac{b_1}{l} + \omega_2 \frac{b_2}{l} + \omega_3 \frac{b_3}{l} + \omega_4 \frac{b_4}{l}\right];$$

$$M_A l + 2M_B(l + l_0) + M_4 l_0 = -6\left[\omega_1 \frac{a_1}{l} + \omega_2 \frac{a_2}{l} + \omega_2 \frac{a_3}{l} + \omega_4 \frac{a_4}{l}\right].$$

We take $l_0 = 0$ and substitute the values of the remaining quantities

$$24M_A + 12M_B = -\frac{6}{12}\left[\left(\frac{243}{2} \times 6\right) + \left(\frac{81}{2} \times 2\right) + \left(\frac{175}{3} \times \frac{26}{3}\right) + \left(\frac{245}{3} \times \frac{14}{3}\right)\right];$$

$$12M_A + 24M_B = -\frac{6}{12}\left[\left(\frac{243}{2} \times 6\right) + \left(\frac{81}{2} \times 10\right) + \left(\frac{175}{3} \times \frac{10}{3}\right) + \left(\frac{245}{3} \times \frac{22}{3}\right)\right];$$

$$2M_A + M_B = -70{\cdot}7; \qquad M_A = -20{\cdot}4\,\text{tm};$$

$$M_A + 2M_B = -80{\cdot}3; \qquad M_B = -29{\cdot}9\,\text{tm}.$$

Making use of the values of the bending moment diagrams in figures (c) and (d), and the values obtained for M_A and M_B

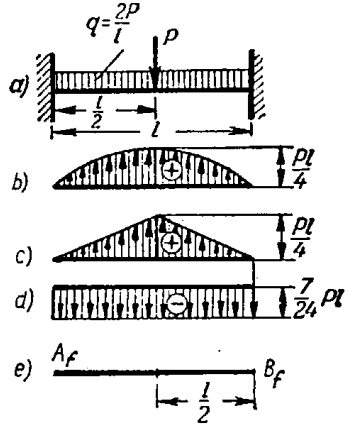
(figure (e)), it will be possible to plot the bending moment diagram by adding the individual diagrams. Alternatively, having determined the reactions at the supports, the bending moment diagram may be plotted by the usual method (figure (f)).

To plot the shear force diagram we determine the reactions

$$-20{\cdot}4 + (A \times 12) - (8 \times 7) - (12 \times 3) = -29{\cdot}9;\ A = 6{\cdot}88\ \text{t};$$

$$-29{\cdot}9 + (B \times 12) - 108 - 40 = 20{\cdot}4;\ B = 13{\cdot}12\ \text{t}.$$

5.151. Determine the load-carrying capacity of a steel No. 30a I-beam with a 6 m span clamped at both ends. It is loaded by a concentrated force P at mid-span, and by a uniformly distributed load $2P$. The allowable stress is 1200 kg/cm². Calculate the deflection at the point of application of the force P.



PROBLEM 5.151

Solution. The fixing moment, due to the concentrated force, is

$$M_A = M_B = -\frac{Pl}{8}.$$

The fixing moment, due to the uniformly distributed load of intensity $q = 2P/l$, is

$$M_A = M_B = -\frac{ql^2}{12} = -\frac{Pl}{6}.$$

Thus the fixing moment, due to the simultaneous action of P and q, is

$$M_A = M_B = -\frac{7}{24} Pl.$$

The maximum positive bending moment (at the point of application of force P) has the value $M_P = 5/24\,Pl$. The condition of limiting strength for the critical section is

$$\frac{7}{24} Pl : W = [\sigma].$$

After substituting $l = 6\,\text{m}$, $W = 597\,\text{cm}^3$ and $[\sigma] = 1200\,\text{kg/cm}^2$, we have

$$P = 12 \times 597 \times \frac{4}{7} = 4090\ \text{kg}.$$

To determine the value of the deflection we shall employ the semi-graphical method. We plot bending moment diagrams (taken as fictitious loads) for P (figure (b)), q (figure (c)) and $M_A = M_B$ (figure (d)). The fictitious beam has no supports (figure (e)). $A_\Phi = B_\Phi = 0$.

We calculate the fictitious bending moment at the mid-span of the beam

$$M_\Phi = \left(\frac{1}{2} \times \frac{Pl}{4} \times \frac{l}{2} \times \frac{1}{6}\right) + \left(\frac{2}{3} \times \frac{Pl}{4} \times \frac{l}{2} \times \frac{3}{8} \times \frac{l}{2}\right) - \left(\frac{7}{24} Pl \times \frac{l}{2} \times \frac{l}{4}\right) = -\frac{Pl^3}{96}.$$

The deflection at the point of application of the force P is

$$f = -\frac{Pl^3}{96EJ} = -\frac{4090 \times 600^3}{96 \times 2 \times 10^6 \times 8950} = -0{\cdot}51\ \text{cm}.$$

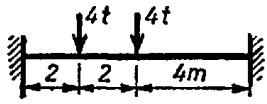
5.152. A beam with a span of 6 m is clamped at both ends and loaded by a concentrated force P at mid-span. The material of the beam is pine ($[\sigma] = 70\,\text{kg/cm}^2$, $E = 10^5\,\text{kg/cm}^2$). The cross-

section is rectangular, with a width of 12 cm, and a depth of 30 cm. Determine the load-carrying capacity of the beam, and the deflection at the point of application of the force P.

Answer: $P = 1{\cdot}68$ t; $f = 0{\cdot}7$ cm.

5.153. Plot M and Q diagrams for the beam shown in the figure. Select an I-section if $[\sigma]$ is 1400 kg/cm^2. Find the deflection at mid-span.

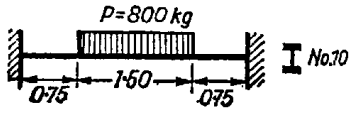
Answer: $M_{max} = 8{\cdot}5$ tm; I No. 30a; 0·895 cm.



PROBLEM 5.153

5.154. Determine the fixing moments at the supports, and the deflection at mid-span of the beam shown in the figure. Construct M and Q diagrams. What is the maximum normal stress in this beam? What is the deflection at mid-span?

Answer: 561 kg/cm^2; 0·186 cm.



PROBLEM 5.154

5.155. A beam with a span of 4 m is clamped at both ends, and loaded by a single concentrated force. The moment at the left-hand support is 3 tm, and that at the right-hand support is 1·5 tm. The section having a maximum deflection is at a distance of 1·8 m from the left-hand support. Determine the bending moment at this section, the maximum bending moment, and the position and magnitude of the load.

Answer: 1·39 tm; 2·02 tm; 5·08 t at a distance of 2·67 m from the right-hand support.

5.156. Determine the fixing moments at the supports of a beam with a span l, clamped at both ends, and loaded by a triangular-

shaped load having zero intensity at one support, and an intensity q at the other support.

Answer: $-\frac{ql^2}{30}$; $-\frac{ql^2}{20}$.

5.157. A beam is clamped at both ends and has also two intermediate supports, dividing the beam into three equal sections, each of length l. Determine the reactions and the fixing moments at the supports due to a uniformly distributed load of intensity q.

Answer: $B = C = ql$; $M_A = M_B = -\frac{ql^2}{12}$.

5.158. A continuous beam AE of constant cross-section is supported at points A, B, C and D. The portion DE is overhung. AB = 3·6 m, BC = 2·4 m, CD = 1·8 m and DE = 1·2 m. A concentrated force of 6 t is applied at a distance of 2·4 m from A, and a uniformly distributed load of intensity 6 t/m is applied to span BC. A concentrated force of 4 t is applied to span CD at a distance of 0·6 m from C; a concentrated force of 5 t is applied at point E. Determine the bending moments at the characteristic points of the beam, and find the reactions at the supports.

Answer: The bending moments (in tm) are +2·13; −4·01; +2·16 (at 5·03 m from A); −0·61; −0·47; −6·0. The reactions (in t) are 0·89; 13·7; 5·17; 9·64.

5.159. A four-span continuous beam with equal spans of 2 m is loaded over the whole length by a uniformly distributed load of intensity of 450 kg/m. Choose the rectangular cross-section in which the depth is twice the width, if $[\sigma]$ is 70 kg/cm².

Answer: 7·5 × 15 cm².

5.160. A four-span continuous beam of I-section (No. 30a) has span lengths of 3 m, 4 m, 4 m and 3 m. All four spans are loaded by a uniformly distributed load, with an intensity of 2·4 t/m on the first and the fourth span, and 3·6 t/m on the second and the third span. Determine the maximum normal stress in the beam.

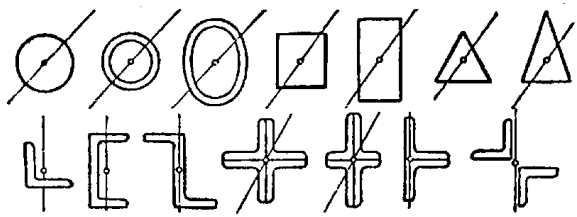
Answer: 894 kg/cm².

CHAPTER 6

COMBINED LOADING

25. UNSYMMETRICAL FLEXURE

6.1. The figure shows a number of beam cross-sections together with the planes of bending moments acting on the beams. Indicate the cases of plane flexure, and the cases of unsymmetrical flexure.



PROBLEM 6.1

6.2. A simply supported timber beam of rectangular cross-section, with a width $b = 15$ cm and a depth $h = 20$ cm, is loaded at the middle of the span, $l = 3$ m, by a concentrated load $P = 1$ t. The plane of action of the load forms an angle $\Phi = 15°$ with the vertical plane passing through the axis of the beam (see figure). Determine the position of the neutral axis, and the maximum compressive stress in the cross-section of the beam. Determine also the total deflection of the mid-span of the beam in magnitude and direction.

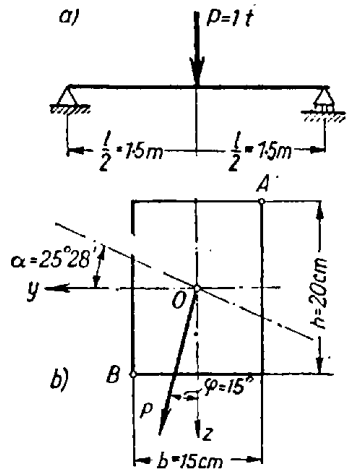
Solution. The tangent of the angle of inclination of the neutral axis to the y axis is found from the formula

$$\tan \alpha = \frac{J_y}{J_z} \tan \Phi = \frac{12bh^3}{12hb^3} \tan \Phi = \frac{h^2}{b^2} \tan \Phi =$$

$$= \frac{20^2}{15^2} \times 0{\cdot}268 = 0{\cdot}476 = \tan 25°28'.$$

The position of the neutral axis is shown in the figure. The critical points are A and B, which are furthermost from the neutral axis. The maximum compressive stress occurs at point A; to calculate it we make use of the formula

$$\sigma_A = -M_{max}\left(\frac{\cos \Phi}{W_y} + \frac{\sin \Phi}{W_z}\right).$$



PROBLEM 6.2

Since $M_{max} = Pl/4$,

$$\sigma_A = -\frac{Pl}{4}\left(\frac{6\cos \Phi}{bh^2} + \frac{6\sin \Phi}{hb^3}\right) = -\frac{3\,Pl}{2bh}\left(\frac{\cos \Phi}{h} + \frac{\sin \Phi}{b}\right)$$

$$= -\frac{3 \times 1000 \times 300}{2 \times 15 \times 20}\left(\frac{0{\cdot}966}{20} + \frac{0{\cdot}259}{15}\right) = -98{\cdot}3 \text{ kg/cm}^2.$$

The maximum deflection of the beam occurs at the mid-span. The components of this deflection in the direction of the y and z axes can be calculated from the formulae

$$f_y = \frac{P_y l^3}{48EJ_z} \quad \text{and} \quad f_z = \frac{P_z l^3}{48EJ_y},$$

where $P_y = P \sin \Phi$ and $P_z = P \cos \Phi$ are the components of the force P along the y and z axes. The total deflection is

$$f = \sqrt{f_y^2 + f_z^2} = \sqrt{\left(\frac{Pl^3 \sin \Phi}{48EJ_z}\right)^2 + \left(\frac{Pl^3 \cos \Phi}{48EJ_y}\right)^2}$$

$$= \frac{Pl^3}{48E} \sqrt{\left(\frac{12 \sin \Phi}{hb^3}\right)^2 + \left(\frac{12 \cos \Phi}{bh^3}\right)^2}$$

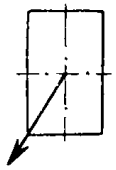
$$= \frac{Pl^3}{4Ebh} \sqrt{\frac{\sin^2 \Phi}{b^4} + \frac{\cos^2 \Phi}{h^4}}$$

$$= \frac{1000 \times 300^3}{4 \times 10^5 \times 20 \times 15} \sqrt{\frac{0{\cdot}259^2}{15^4} + \frac{0{\cdot}966^2}{20^4}} = 0{\cdot}605 \text{ cm}.$$

The direction of the deflection will form an angle $\alpha = 25°28'$ with the direction of the z axis.

6.3. What position will the neutral layer in a beam of rectangular cross-section assume, if the plane of action of the load coincides with one of the diagonal planes (see figure)?

Answer: The neutral layer will coincide with the other diagonal plane.

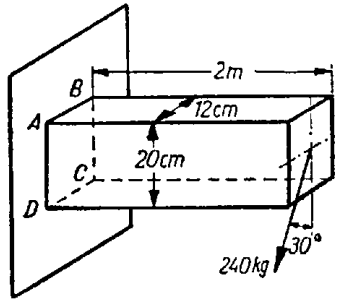


PROBLEM 6.3

6.4. A wooden cantilever 2 m long, having a rectangular cross-section of 12 × 20 cm², is loaded by a concentrated load of 240 kg at the free end. The load lies in a transverse plane of the beam, and passes through its centre of gravity (see figure).

Plot a diagram showing the variation in the normal stress along the sides of the fixed end, and determine the total deflection of the free end of the beam.

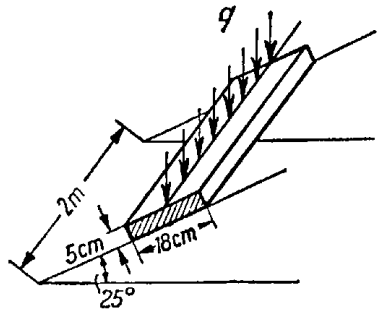
Answer: $\sigma_A = +2$ kg/cm^2; $\sigma_B = +102$ kg/cm^2; $\sigma_C = -2$ kg/cm^2; $\sigma_D = -102$ kg/cm^2; $f = 1{\cdot}31$ cm.



PROBLEM 6.4

6.5. A roof is inclined at an angle of 25° to the horizontal. The purlins, in the form of 5 × 18 cm^2 planks with a 2 m span, are simply supported and loaded by a vertical uniformly distributed load of an intensity q (see figure). Determine the maximum allowable value of this load, if $[\sigma]$ is 100 kg/cm^2, and calculate the total deflection of the purlin at its mid-span.

Answer: 146 kg/m; 1·47 cm.



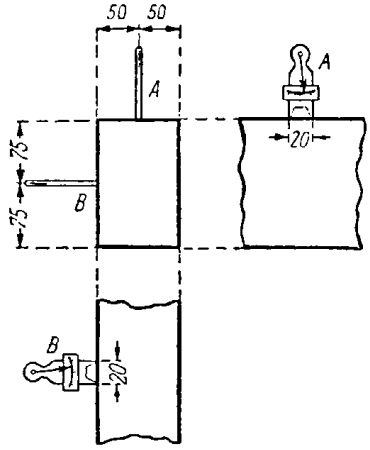
PROBLEM 6.5

6.6. A simply supported timber beam 2·4 m long, with a rectangular cross-section of 10 × 15 cm^2, is loaded at the mid-span by a concentrated force P.

Extensometers A and B, having a gauge length of 20 mm and a magnification of 1000, were placed at the critical section of the beam (see figure). Under the load a decrease of 7 mm at A, and an increase of 10 mm at B were noted in the extensometer readings.

Determine the magnitude and the direction of the force P, and also the value of the maximum normal stress in the beam.

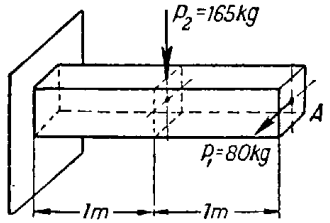
Answer: $P = 302$ kg; $\Phi = 43°36'$; $\sigma_{max} = 85$ kg/cm^2.



PROBLEM 6.6

6.7. A timber cantilever 2 m long is bent by forces P_1 and P_2 (see figure). Choose the rectangular cross-section for the beam, with the ratio of the depth to the width equal to 2, and determine the total deflection at section A in magnitude and direction. The depth of the beam is parallel to the plane of action of the force P_2. The allowable stress $[\sigma]$ is 100 kg/cm^2.

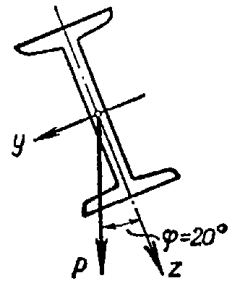
Answer: 9×18 cm^2; $f_A = 1{\cdot}98$ cm (at an angle of $80°52'$ to the vertical).



PROBLEM 6.7

6.8. A vertical force $P = 700$ kg is applied to a simply supported I-beam at the middle of the span l, of 4 m. The plane of the web

of the I-section forms an angle of 20° with the vertical (see figure). Select the cross-section for the beam from the tables in Appendix A, if the allowable stress $[\sigma]$ is 1600 kg/cm².



PROBLEM 6.8

Solution. The condition of limiting strength in the given case (a cross-section with protruding corners) has the form

$$\sigma_{\max} = M_{\max}\left(\frac{\cos\Phi}{W_y} + \frac{\sin\Phi}{W_z}\right) = \frac{Pl}{4W_y}\left(\cos\Phi + \frac{W_y}{W_z}\sin\Phi\right) \leq [\sigma],$$

whence

$$W_y \geq \frac{Pl}{4[\sigma]}\left(\cos\Phi + \frac{W_y}{W_z}\sin\Phi\right)$$

$$= \frac{700 \times 400}{4 \times 1600}\left(0{\cdot}940 + \frac{W_y}{W_z} \times 0{\cdot}342\right)$$

$$= 41{\cdot}08 + 14{\cdot}96\,\frac{W_y}{W_z}.$$

We find the section by means of trial and error. The ratio W_y/W_z for all sizes of standard I-beams varies approximately from 6 (for the smallest sizes) to 15 (for the largest sizes). If we assume that $W_y/W_z = 6$, then $W_y \geq 41{\cdot}08 + (14{\cdot}96 \times 6) \approx 130$ cm³; if however $W_y/W_z = 15$, then $W_y \geq 41{\cdot}08 + (14{\cdot}96 \times 15) \approx 265$ cm³. The section modulus W_y for a No. 20b I-section is 250 cm³; the section modulus W_y for a No. 16 I-section is 141 cm³. Thus, the required I-section must presumably be contained between the sizes 20b and 16. We try a No. 18 I-section for which $W_y = 185$ cm³,

$W_z = 26\ \text{cm}^3$ and $W_y/W_z = 185/26 = 7{\cdot}12$. We calculate the maximum normal stress

$$\sigma_{\max} = \frac{Pl}{4W_y}\left(\cos\Phi + \frac{W_y}{W_z}\sin\Phi\right)$$

$$= \frac{700 \times 400}{4 \times 185}(0{\cdot}94 + 7{\cdot}12 + 0{\cdot}342) = 1275\,\text{kg/cm}^2.$$

The maximum stress obtained is considerably lower than the allowable value. A smaller I-section ought to be taken. We try a No. 16 I-section, for which

$$W_y = 141\ \text{cm}^3; \quad W_z = 21{\cdot}2\ \text{cm}^3 \quad \text{and} \quad \frac{W_y}{W_z} = \frac{141}{21{\cdot}2} = 6{\cdot}65.$$

The maximum normal stress is

$$\sigma_{\max} = \frac{700 \times 400}{4 \times 141}[0{\cdot}94 + (6{\cdot}65 \times 0{\cdot}342)]$$

$$= 1596\ \text{kg/cm}^2 \approx [\sigma] = 1600\ \text{kg/cm}^2.$$

Thus I-section No. 16 is chosen.

6.9. An error was made while installing a No. 60a I-beam designed to work in flexure in the vertical plane coinciding with the web of the section. As a result the web deviated from the vertical by an angle Φ of 1°.

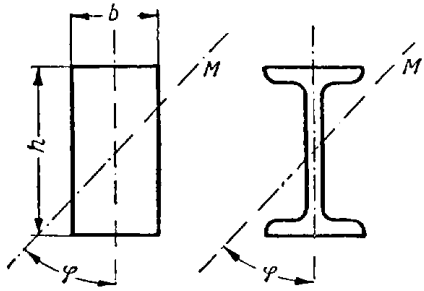
Determine the increase in the maximum normal stress, and that in the total vertical deflection of the I-beam.

Answer: The stress has increased by 25·3%; the total deflection has increased by 31·9%.

6.10. For what angle of inclination (to the vertical plane) of the plane of the bending moment (M) (see figure) is the load-carrying capacity of beams a minimum?

Answer: For an angle Φ for which $\tan\Phi = W_{\max}/W_{\min}$. For a rectangle $\tan\Phi = h/bn$; thereby the load-carrying capacity of the beam is $\sqrt{1 + 1/n^2}$ times less than its load-carrying capacity in plane flexure in the horizontal plane. For I-sections the angle Φ varies from 78°47′ (No. 10) to 86°8′30″ (No. 60c); thereby the

load-carrying capacity is reduced by 1·95% (No. 10) and 0·23% (No. 60c) respectively.

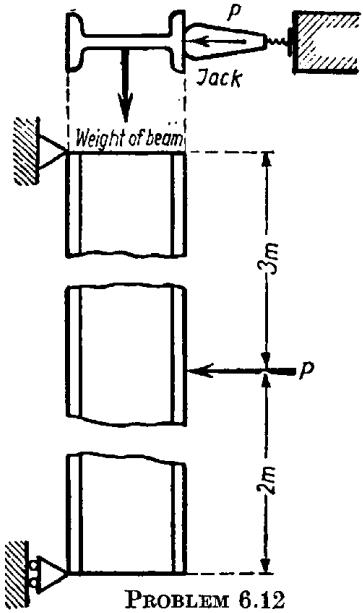


PROBLEM 6.10

6.11. A No. 24a I-section cantilever 1·4 m long, loaded at the free end by a concentrated force P, has deflected 3 mm under the force. The direction of the deflection has coincided with the bisector of the angle between the principal axes of the beam cross-section. Determine the magnitude and the direction of the force P, and also the maximum tensile stress in the beam.

Answer: $P = 2125$ kg; $\Phi = 3°30'$; $\sigma_{max} = 1155$ kg/cm^2.

6.12. A pin-ended No. 40a I-beam of a 5 m span, with the web horizontal, is used provisionally as the base for a horizontal jack



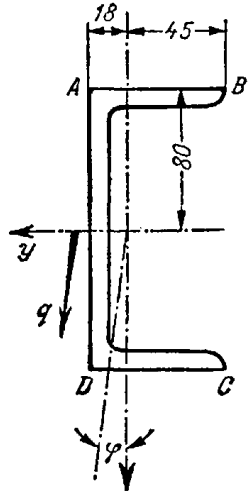
PROBLEM 6.12

located at a distance of 2 m from one of the beam supports (see figure). Taking into acc o unt the weight of the beam, determine the maximum allowable load on the jack. The allowable stress $[\sigma]$ for the material of the beam is 1600 kg/cm^2.

Answer: 12·5 t.

6.13. A cantilever 2 m long carries a uniformly distributed load having an intensity $q = 600$ kg/m. The cross-section of the beam is a No. 16a channel. The web of the channel is inclined at an angle $\Phi = 3°$ to the plane of action of the load, which passes through the shear centre.

Determine the normal stress at points A, B, C, and D (see figure). of the critical section of the beam.



PROBLEM 6.13

Solution. We determine the normal stress in any point of the critical cross-section of the beam, using the formula

$$\sigma = -M_{\max}\left(\frac{y \sin \Phi}{J_z} + \frac{z \cos \Phi}{J_y}\right).$$

The minus sign is placed in front of $M_{\max}$ because the normal stress is compressive, i.e. negative in the quadrant having positive values of y and z co-ordinates.

Since $M_{max} = (ql)^2/2 = (6 \times 200^2/2) = 120{,}000$ kg cm, $\sin \Phi = \sin 3° = 0{\cdot}0532$, $\cos \Phi = \cos 3° = 0{\cdot}9986$, and for a channel No. 16a $J_y = 866{\cdot}2$ cm^4, and $J_z = 73{\cdot}3$ cm^4, we have

$$\sigma = -120{,}000 \left(\frac{0{\cdot}0532}{73{\cdot}3} y + \frac{0{\cdot}9986}{866{\cdot}2} z \right) = -87{\cdot}1y - 138{\cdot}3z.$$

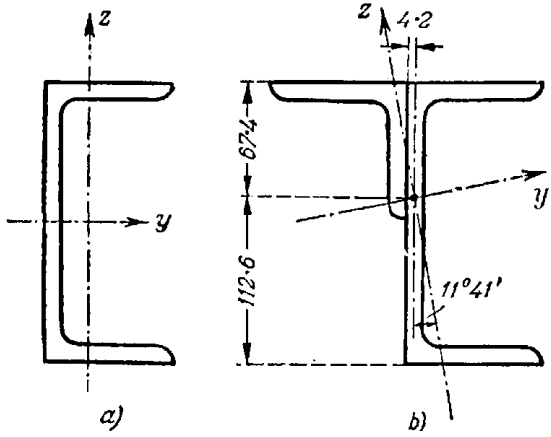
At point A, with the co-ordinates $y_A = +1{\cdot}8$ cm and $z_A = -8$ cm,

$$\sigma_A = -(87{\cdot}1 \times 1{\cdot}8) + (138{\cdot}3 \times 8) = 950 \text{ kg/cm}^2.$$

Similarly

$$\sigma_B = (87{\cdot}1 \times 4{\cdot}5) + (138{\cdot}3 \times 8) = 1498 \text{ kg/cm}^2;$$
$$\sigma_C = (87{\cdot}1 \times 4{\cdot}5) - (138{\cdot}3 \times 8) = -714 \text{ kg/cm}^2;$$
$$\sigma_D = -(87{\cdot}1 \times 1{\cdot}8) - (138{\cdot}3 \times 8) = -1263 \text{ kg/cm}^2.$$

6.14. A simply supported beam carries a concentrated load P at the middle of the span 3 m long. The cross-section of the beam is a No. 18b channel (figure (a)). An additional 80 mm $\times$ 80 mm $\times$ $\times$ 10 mm angle is to be welded to the web of the channel over its full length (figure (b)). The position of the principal centroidal y and z axes of the proposed cross-section is shown in figure (b). The second moments of area about these axes are J_y of 1967 cm^4 and J_z of 306 cm^4.



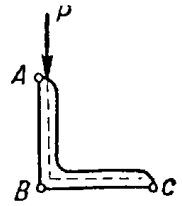
PROBLEM 6.14

Determine, for an allowable stress $[\sigma]$ of 1600 kg/cm², the maximum load-carrying capacity of (a) the channel alone, and (b) the channel with the angle welded to it. Assume that the plane of action of the load in both cases is parallel to the plane of the web of the channel and passes through the centre of shear of the cross-section.

Answer: (a) 3250 kg; (b) 2740 kg; the addition of the angle reduces the load-carrying capacity of the beam by nearly 16%.

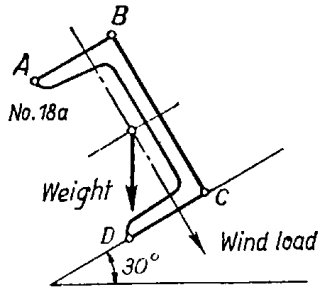
6.15. A 200 mm × 200 mm × 20 mm angle with a length of 4 m works as a simply supported beam. At the mid-span it is loaded by a concentrated force P of 2500 kg, the direction of which passes through the shear centre of the cross-section (see figure). Determine the normal stress at points A, B and C in the critical cross-section.

Answer: $\sigma_A = -1463$ kg/cm²; $\sigma_B = +1213$ kg/cm²;
$\sigma_C = -367$ kg/cm².



PROBLEM 6.15

6.16. The rafters of a roof are 3 m apart and inclined to the horizontal at an angle of 30°. Channel No. 18a purlins, located at a distance of 2 m from each other, are fixed to the rafters. A wind



PROBLEM 6.16

load, directed perpendicularly to the plane of the roof, has an intensity of 100 kg/m², whereas the weight of the roofing, acting vertically, has an intensity of 75 kg/m². Determine the normal stress at points A, B, C, and D in the critical section (see figure). Do not take torsion of the channel into consideration.

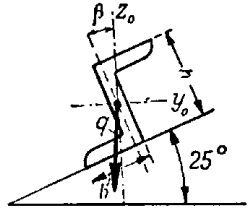
Answer: $\sigma_A = +159$ kg/cm²; $\sigma_B = -423$ kg/cm²;

$\sigma_C = +102$ kg/cm²; $\sigma_D = +683$ kg/cm².

6.17. A roof purlin of Z-section No. 14 works as a simply supported beam 2·5 m in span, loaded by a uniformly distributed load with an intensity q. The plane of action of the load passes through the shear centre of the cross-section (see figure). The position of the principal centroidal axes of the cross-section is shown in the figure. $\beta = 21°48'$; $J_{y0} = 847$ cm⁴, $J_{z0} = 61{\cdot}4$ cm⁴, $b = 65$ mm, $h = 140$ mm.

Determine the maximum allowable value of the load, if the allowable stress $[\sigma]$ is 1600 kg/cm², and find the magnitude and the direction of the total deflection of the purlin at the mid-span due to this load.

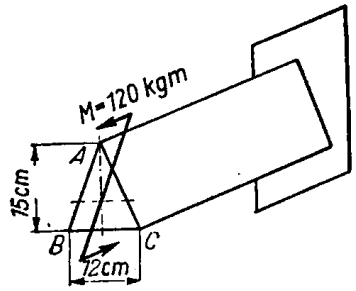
Answer: 1·5 t/m; 0·57 cm; 34°26′ to the vertical.



PROBLEM 6.17

6.18. A beam of triangular cross-section (see figure) is bent by a moment $M = 120$ kgm in the plane parallel to side AB. Determine the position of the neutral axis, and the stress at the vertices of the triangle. Determine also the maximum value of the bending moment M when the beam is bent in the vertical plane. $[\sigma] = 100$ kg/cm².

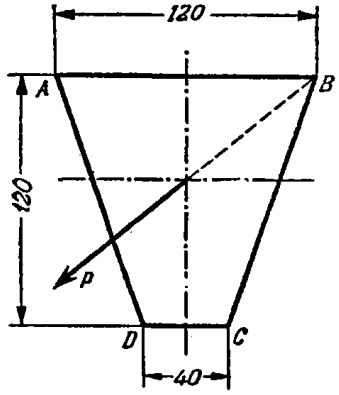
Answer: $\alpha = 39°48'$; $\sigma_A = -\sigma_B = 99\ \text{kg/cm}^2$; $\sigma_C = 0$; $M = 112{\cdot}5\ \text{kgm}$.



PROBLEM 6.18

6.19. A cast iron beam with a cross-section in the form of a trapezium (see figure) has a length of 1·6 m. It is simply supported and loaded at the middle of the span by a concentrated force P, the direction of which passes through the centroid of the cross-section, and through point B. Determine the maximum allowable value of the load P, and the magnitude of the normal stress due to this load at the vertices of the trapezium. The allowable stress for cast iron is 400 kg/cm² in tension, and 1200 kg/cm² in compression.

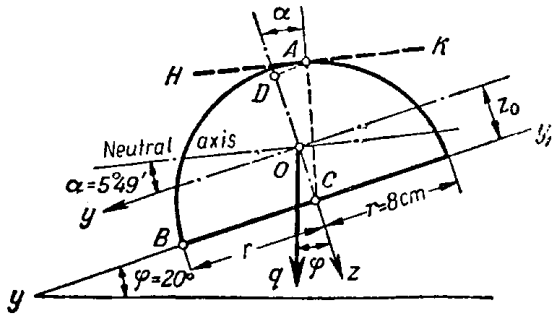
Answer: $P \approx 1500\,\text{kg}$; $\sigma_C = +111\,\text{kg/cm}^2$; $\sigma_D = +399\,\text{kg/cm}^2$; $\sigma_A = +250\ \text{kg/cm}^2$; $\sigma_B = -614\ \text{kg/cm}^2$.



PROBLEM 6.19

6.20. A simply supported beam with a semicircular cross-section 16 cm in diameter, having a length of 2 m, is loaded by a uniformly

distributed load with an intensity of $q = 200$ kg/m (see figure). Determine the maximum tensile and compressive stresses in the beam.



PROBLEM 6.20

Solution. The centroid of the cross-section of the beam is at point O, at a distance $z_0 = 4r/3\pi = (4 \times 8)/(3 \times 3{\cdot}14) = 3{\cdot}4$ cm from the diametral y_1 axis. The second moments of area of the cross-section about the principal centroidal axes y and z are respectively

$$J_z = \frac{1}{2}\,\frac{\pi r^4}{4} = \frac{3{\cdot}14 \times 8^4}{8} = 1608 \text{ cm}^4$$

and

$$J_y = J_{y1} - z_0^2 F = \frac{\pi r^4}{4} - z_0^2\,\frac{\pi r^4}{2} = \frac{3{\cdot}14 \times 8^4}{8}$$

$$- \left(3{\cdot}4^2 \times \frac{3{\cdot}14 \times 8^2}{2}\right) = 450 \text{ cm}^4.$$

The angle of inclination of the neutral axis to the y axis is determined by the formula

$$\tan \alpha = \frac{J_y}{J_z} \tan \Phi = \frac{J_y}{J_z} \tan 20° = \frac{450}{1608} \times 0{\cdot}364 = 0{\cdot}1019.$$

Hence $\alpha = 5°49'$; $\cos \alpha = 0{\cdot}955$, $\sin \alpha = 0{\cdot}1013$. The maximum normal stresses occur at the points furthermost from the neutral axis. These will be points B and A (the point at which the line HK, parallel to the neutral axis, touches the outline of the cross-section).

The normal stress in the critical section of the beam (at the middle of its span) is determined by the formula

$$\sigma = M_{\max}\left(\frac{\cos \Phi}{J_y} z + \frac{\sin \Phi}{J_z} y\right).$$

Since

$$M_{\max} = \frac{ql^2}{8} = \frac{200 \times 2^2}{8} = 100 \text{ kgm},$$

then

$$\sigma = 10{,}000\left(\frac{0{\cdot}940}{450} z + \frac{0{\cdot}342}{1608} y\right) = 20{\cdot}89z + 2{\cdot}13y \text{ (kg/cm}^2\text{)}.$$

The normal stress σ_B at point B, with the co-ordinates $z_B = z_0 = 3{\cdot}4$ cm and $y_B = r = 8$ cm, is tensile; thus $\sigma_B = 20{\cdot}89z_B + 2{\cdot}13y_B = (20{\cdot}89 \times 3{\cdot}4) + (2{\cdot}13 \times 8) = 88 \text{ kg/cm}^2$. The coordinates of point A are

$$|z_A| = \overline{OD} = \overline{CD} - \overline{OC} = r\cos\alpha - z_0$$

$$= (8 \times 0{\cdot}995) - 3{\cdot}4 = 4{\cdot}56 \text{ cm}$$

and

$$|y_A| = \overline{AD} = \overline{AC}\sin\alpha = r\sin\alpha = 8 \times 0{\cdot}1013 = 0{\cdot}81 \text{ cm}.$$

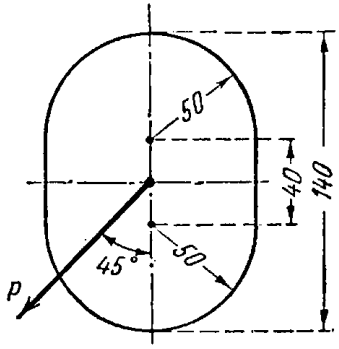
Both co-ordinates are negative, and the stress at point A will be compressive. Thus

$$\sigma_A = 20{\cdot}89z_A + 2{\cdot}13y_A$$

$$= -(20{\cdot}89 \times 4{\cdot}56) - (2{\cdot}13 \times 0{\cdot}81) = -97 \text{ kg/cm}^2.$$

6.21. A wooden beam with the cross-section shown in the figure is simply supported. A concentrated force P, directed at an angle of 45° to the axis of symmetry of the cross-section, is applied to the beam at the mid-span. The span of the beam is 3 m. Determine the maximum allowable value of the force P, if the allowable stress is 100 kg/cm².

Answer: 222 kg.

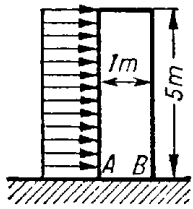


PROBLEM 6.21

26. FLEXURE COMBINED WITH TENSION OR COMPRESSION

6.22. A brickwork pillar with a 1×1 m^2 square cross-section, 5 m high, is loaded by its own weight and a transverse uniformly distributed wind pressure of 80 kg/m^2 (see figure). The specific gravity of the brickwork of the pillar is 1·6. Determine the maximum and minimum compressive stresses at the base of the pillar.

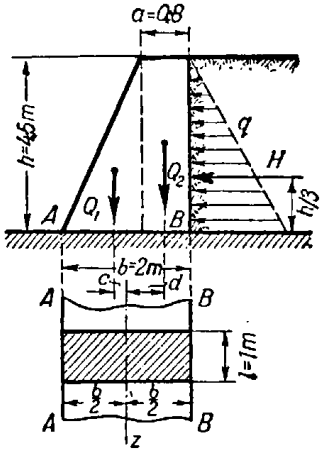
Answer: $\sigma_A = -2$ t/m^2; $\sigma_B = -14$ t/m^2.



PROBLEM 6.22

6.23. A masonry retaining wall, the transverse section of which is diagrammatically represented in the figure, supports an earth embankment. The specific gravity of the masonry is 1·8. The earth pressure q is directed horizontally, and varies linearly along the height of the wall; the maximum pressure at the base of the wall, is 1·5 t/m^2, and the pressure at the top is zero.

Determine the maximum and minimum compressive stresses at the base of the wall.



PROBLEM 6.23

Solution. We isolate a portion, with a length of 1 m, from the wall. This portion may be considered as a cantilever, clamped at its lower end, being bent by the earth pressure, and compressed by its own weight. The cross-section at the base will be stressed most. The maximum and minimum compressive stresses in this cross-section may be calculated by the formula

$$\sigma_{\min}^{\max} = \frac{N}{F} \pm \frac{M}{W}.$$

The force N is the weight of the isolated portion of the wall; $N = Q_1 + Q_2$, where Q_1 is the weight of the triangular prism, and Q_2 the weight of the rectangular prism. Since

$$Q_1 = \frac{1}{2} h(b - a) l\gamma = \frac{1}{2} \times 4{\cdot}5\,(2 - 0{\cdot}8) \times 1 \times 1{\cdot}8 = 4{\cdot}86\ \text{t},$$

and

$$Q = hal\gamma = 4{\cdot}5 \times 0{\cdot}8 \times 1 \times 1{\cdot}8 = 6{\cdot}48\ \text{t},$$

then

$$N = 4{\cdot}86 + 6{\cdot}48 = 11{\cdot}34\ \text{t}.$$

The bending moment M is equal to the sum of the moments of the weights Q_1 and Q_2 and the resultant of the earth pressure H about the centroidal z axis of the base of the wall:

$$M = Q_1\left(a + \frac{b-a}{3} - \frac{b}{2}\right) - Q_2\left(\frac{b}{2} - \frac{a}{2}\right) + \frac{q_{\max} h}{2}\,\frac{h}{3}$$

$$= (4{\cdot}86 \times 0{\cdot}2) - (6{\cdot}48 \times 0{\cdot}6) + \frac{1{\cdot}5 \times 4{\cdot}5}{2}\,\frac{4{\cdot}5}{3} = 2{\cdot}15\,\text{tm}.$$

Since $F = bl = 2 \times 1 = 2\,\text{m}^2$ and $W = W_z = (lb^2/6) = (1 \times 2^2/6) = 0{\cdot}67\,\text{m}^3$, then

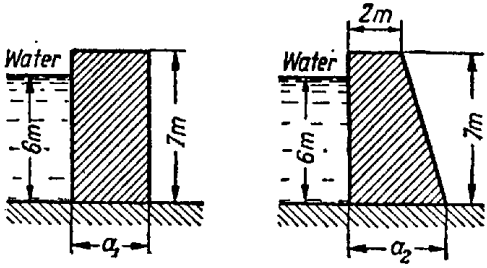
$$\sigma_{\min}^{\max} = \frac{11{\cdot}34}{2} \pm \frac{2{\cdot}15}{0{\cdot}67} = 5{\cdot}67 \pm 3{\cdot}22\,\text{t/m}^2.$$

Thus

$$\sigma_{\max} = \sigma_A = 8{\cdot}89\,\text{t/m}^2 \quad \text{and} \quad \sigma_{\min} = \sigma_B = 2{\cdot}45\,\text{t/m}^2.$$

6.24. Two versions of a transverse section of a concrete dam 7 m in height are represented diagrammatically in the figure. Assuming that the specific gravity of the concrete is 2, determine the required widths a_1 and a_2 of the dam so that no tensile stresses would be set up in its base.

Answer: $a_1 = 3{\cdot}93$ m; $a_2 = 3{\cdot}52$ m.



PROBLEM 6.24

6.25. A factory chimney in the form of a hollow truncated cone has a height of 32 m. The external diameter of the top section of the chimney is 2 m, the external diameter of the base is 3·5 m; the

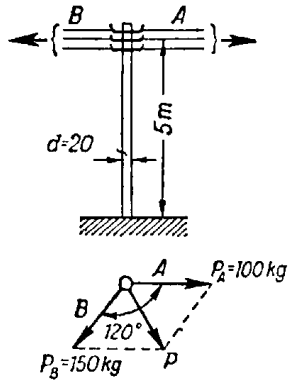
wall thickness is correspondingly 25 cm and 1 m. The weight of 1 m^3 of the brickwork is 1·6 t.

Determine the maximum value of the wind load on 1 m^2 of the vertical projected area of the chimney, for which there will be no tensile stresses in the chimney base. Assume that the wind pressure along the diametral section of the chimney is uniformly distributed.

Answer: 89·5 kg/m^2.

6.26. A post 20 cm in diameter carries, at a height of 5 m above the ground, two groups of horizontal wires: A and B, the angle between of which is 120° (see figure). The horizontal force taken by the post from group A wires is 100 kg; from group B wires it is 150 kg. The weight transmitted to the post from the wires of both groups is 280 kg; the weight of the post is 90 kg. Determine the maximum tensile stress, and the maximum compressive stress in the cross-section of the post at the ground level.

Answer: The maximum tensile stress $\max \sigma_t = 83$ kg/cm^2, the maximum compressive stress $\max \sigma_c = 85{\cdot}4$ kg/cm^2.

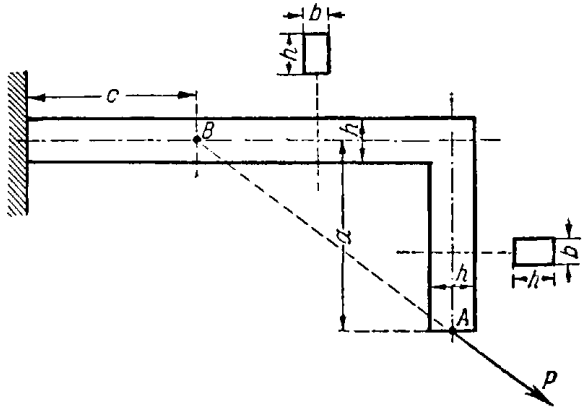


PROBLEM 6.26

6.27. A cranked rod of rectangular cross-section (see figure) is loaded by a force P, the direction of which passes through the centroids A and B of the cross-section. At what point will the normal stress be the greatest?

What will it be, if $P = 2$ t, $h = 10$ cm, $b = 6$ cm, $a = 90$ cm, $c = 80$ cm, and the length l of the horizontal crank is 2 m?

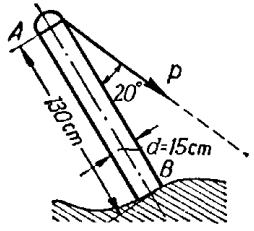
Answer: 1407 kg/cm².



PROBLEM 6.27

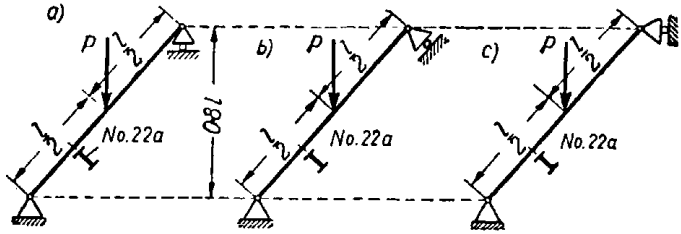
6.28. A wooden post AB of circular cross-section, 15 cm in diameter (see figure), holds a rope which is being pulled by a force P. Assuming the post clamped at the section B, determine the maximum possible tension in the rope, if the allowable stress for the material of the post is 100 kg/cm².

Answer: 622 kg.



PROBLEM 6.28

6.29. An inclined No. 22a I-beam, with a length $l = 2{\cdot}5$ m, is loaded at the middle of the span by a concentrated force P (= 10 t)



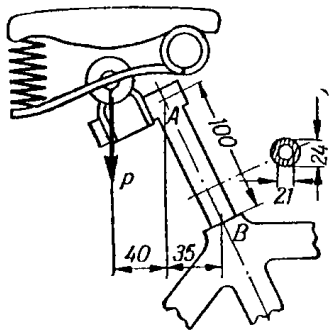
PROBLEM 6.29

directed vertically (see figure). Plot diagrams showing the variation in the bending moment and the axial force along the length of the beam. Determine the value of the maximum compressive stress in the beam for each arrangement of the upper hinged support.

Answer: (a) 1309 kg/cm²; (b) 1404 kg/cm²; (c) 1458 kg/cm².

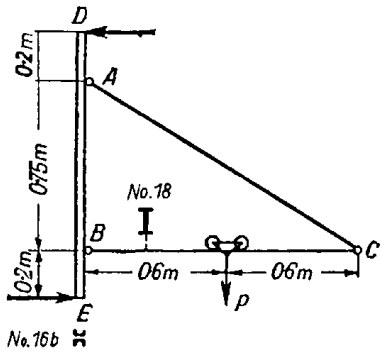
6.30. Determine the normal stress in tube AB carrying a bicycle saddle (see figure), if the load P is 80 kg. The external diameter of the tube is 24 mm; the internal diameter is 21 mm. Assume that the end B of the tube is clamped.

Answer: 1139 kg/cm².



PROBLEM 6.30

6.31. The I-beam BC (I No. 18) of a wall crane is pin-jointed to the upright DE and the tie AC (see figure). Determine the maximum



PROBLEM 6.31

normal stress in the cross-section of the beam, if a load P of 4 t is applied to the mid-span of the beam.

Answer: 754 kg/cm².

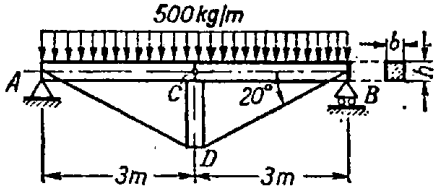
6.32. The upright DE of the wall crane (see the figure to Problem 6.31) is made up of two No. 16 b channels joined back to back. The plane of the backs is parallel to the plane of the figure. Determine the maximum compressive stress in the cross-section of the upright located directly under point A. Assume that the force P (= 4 t) is located at point C.

Answer: 437 kg/cm².

6.33. A simply supported trussed timber beam of rectangular cross-section (with the ratio $h/b = 3/2$) 6 m long is loaded by a uniformly distributed load of 500 kg/m (see figure).

Determine the dimensions of the cross-section of the beam, if the allowable stress is 100 kg/cm², and find the diameter of the steel ties AD and DB for an allowable stress of 1200 kg/cm². In the calculation assume that the beam AB has a hinge at section C.

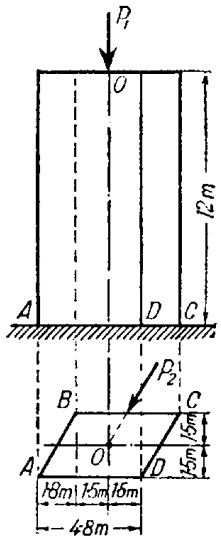
Answer: $h = 17{\cdot}8$ cm; $b = 11{\cdot}9$ cm; $d = 15{\cdot}3$ mm.



PROBLEM 6.33

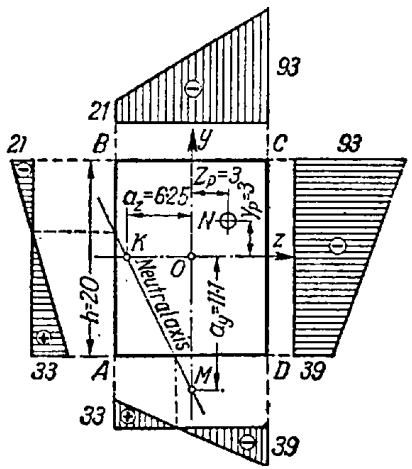
6.34. A masonry pier of a bridge having an oblique angle in plan (see figure) is loaded at the top section by a vertical force P_1 of 110 t (the weight of the bridge span and that of a train passing along it) applied to point O, and a horizontal force P_2 of 20 t (the braking force of the train) parallel to the sides AB and CD of the cross-section of the pier. The height of the pier is 12 m; the specific weight of the masonry is 2·2 t/m³. Determine the maximum and minimum compressive stresses in the horizontal cross-section of the pier at its base.

Answer: $\sigma_{max} = \sigma_A = 64$ t/m²; $\sigma_{min} = \sigma_C = 4{\cdot}1$ t/m².



PROBLEM 6.34

6.35. A wooden post, with a 15 × 20 cm² cross-section is loaded by a longitudinal compressive force of 9 t applied to point N, at a distance of 3 cm from each of the principal axes of the cross-section (see figure). Determine the position of the neutral axis, and plot diagrams showing the variation in the normal stress along the sides of the rectangle.



PROBLEM 6.35

Solution. To find the stress in the cross-section of the post we make use of the formula

$$\sigma = \frac{P}{F}\left(1 + \frac{y_P y}{i_z^2} + \frac{z_P z}{i_y^2}\right).$$

In the given case the force $P = -9000$ kg (the force compressing the post), $F = bd = 15 \times 20 = 300$ cm^2, $y_P = z_P = 3$ cm. The squares of the radii of gyration of the cross-section are

$$i_y^2 = \frac{J_y}{F} = \frac{hb^3}{12bh} = \frac{b^2}{12} = \frac{15^2}{12} = \frac{75}{4} \text{ cm}^2$$

and

$$i_z^2 = \frac{h^2}{12} = \frac{20^2}{12} = \frac{100}{3} \text{ cm}^2.$$

Substituting the values of P, F, y_P, z_P, i_y^z, and i_z^2 in the formula for the stress, we obtain

$$\sigma = -30(1 + 0{\cdot}09y + 0{\cdot}16z).$$

We find the position of the neutral axis by putting in the preceding expression $\sigma = 0$. We have then

$$1 + 0{\cdot}09y_0 + 0{\cdot}16z_0 = 0,$$

whence, at $z_0 = 0$, we have

$$a_y = -\frac{1}{0{\cdot}09} = -11{\cdot}1 \text{ cm} \quad \text{(point M)},$$

and at $y_0 = 0$,

$$a_z = -\frac{1}{0{\cdot}16} = -6{\cdot}25 \text{ cm} \quad \text{(point K)}.$$

The neutral axis passes through points M and K. To plot the diagrams showing the variation in the normal stress along the sides

of the cross-section, it is sufficient to calculate the value of the stress at points A, B, C and D (the corner points of the rectangle), since the stress varies linearly with the y and z co-ordinates. At point A ($y_A = -10$ cm, $z_A = -7{\cdot}5$ cm)

$$\sigma_A = -30\,[1 - (0{\cdot}09 \times 10) - (0{\cdot}16 \times 7{\cdot}5)]$$

$$= +33 \text{ kg/cm}^2 \text{ (tension)}.$$

Similarly

at point B $\sigma_B = -21$ kg/cm² (compression);

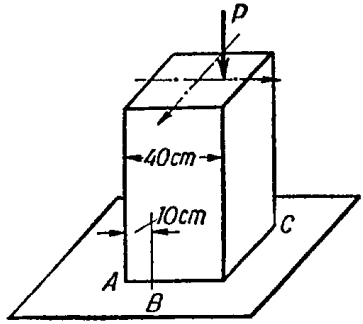
at point C $\sigma_C = -93$ kg/cm² (compression);

at point D $\sigma_D = -39$ kg/cm² (compression).

The required stress diagrams are constructed using these values of σ_A, σ_B, σ_C, and σ_D.

6.36. The normal stress at point A of the bar under a load (see figure) is 20 kg/cm² (tension), at point B it is zero. What is the stress at point C?

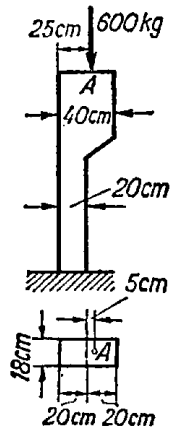
Answer: 60 kg/cm².



PROBLEM 6.36

6.37. Check the strength of the lower portion of a concrete pillar having a 18×20 cm² cross-section (see figure). The allowable stress is 6 kg/cm² in tension, and 70 kg/cm² in compression.

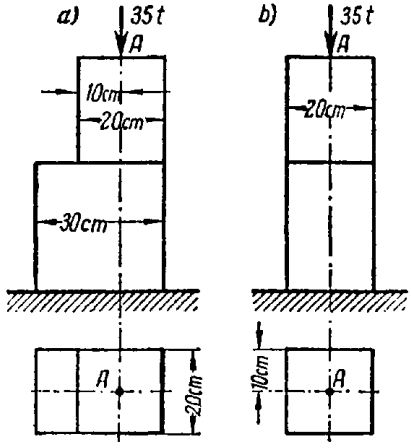
Answer: max $\sigma_t = 5{\cdot}8$ kg/cm^2 < 6 kg/cm^2; max $\sigma_c = 9{\cdot}2$ kg/cm^2 < 70 kg/cm^2.



PROBLEM 6.37

6.38. The foundation for a stand having a cross-section of 20 $\times$ 20 cm^2 is made of a block 20 $\times$ 30 cm^2 in cross-section (figure (a)). Determine the maximum normal stress in the foundation of the stand, if the force compressing it is 35 t. What will be this stress, if the foundation of the stand is made of a block 20 $\times$ 20 cm^2 in cross-section (figure (b))?

Answer: 116·7 kg/cm^2; 87·5 kg/cm^2.



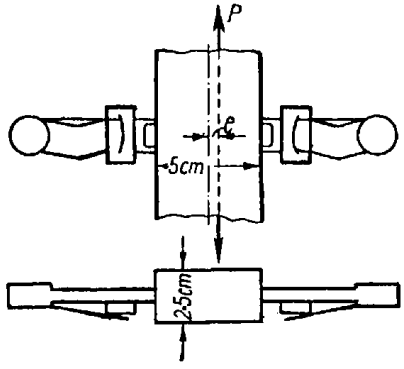
PROBLEM 6.38

6.39. Determine the thickness of a steel strip, 16 cm wide, which s being pulled by two axial forces of 7 t applied to the mid-point of the thickness, at a distance of 4 cm from one edge of the strip. $[\sigma] = 1400$ kg/cm^2.

Answer: 7·8 mm.

6.40. While testing in tension a bar with a $2{\cdot}5 \times 5$ cm^2 cross-section, it was established by means of extensometers (see figure) that for a certain load the stresses on opposite sides were $+300$ kg/cm^2 and $+1200$ kg/cm^2 respectively. Determine the value and the eccentricity of the force pulling the bar, and find the value of the bending moment.

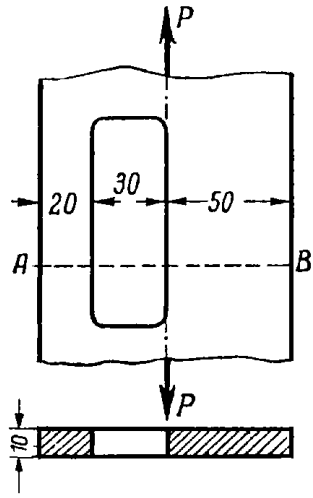
Answer: 9375 kg; 0·5 cm; 4688 kg cm.



PROBLEM 6.40

6.41. A steel strip 100 mm wide and 10 mm thick, being pulled axially by forces $P = 7$ t, has a cut-out 30 mm wide (see figure). Neglecting stress concentrations, draw a diagram showing the variation of the normal stress in section AB. In calculating the second moment of area assume that both parts of the cross-section are rigidly connected to each other. What may be the width of the cut-out, for the same value of the maximum tensile stress, if it is located at the middle of the width of the strip?

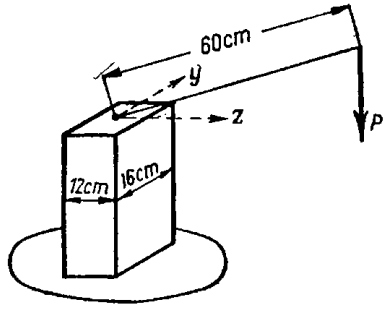
Answer: $\sigma_A = 1355$ kg/cm^2; $\sigma_B = 725$ kg/cm^2; 48·3 mm.



PROBLEM 6.41

6.42. A cantilever 60 cm long is fixed to a short wooden post having a 12 × 16 cm^2 cross-section. It passes through the centroid, and through one corner of the rectangle. A load P is suspended from the end of the cantilever (see figure). Determine the maximum allowable value of the load, if $[\sigma]$ is 100 kg/cm^2.

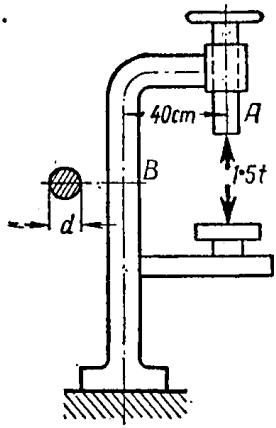
Answer: 519 kg.



PROBLEM 6.42

6.43. An axial force of 1500 kg is transmitted to the spindle of a drilling machine during the drilling of a component. Determine the diameter of the circular cast-iron column, if the allowable stress in tension is 350 kg/cm^2.

Answer: 122 mm.



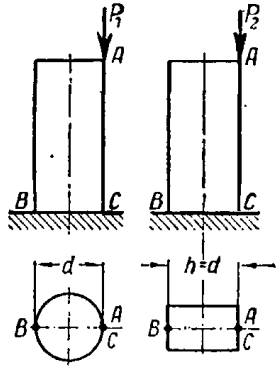
PROBLEM 6.43

6.44. A hollow circular bar, having an external diameter of 75 mm and an internal diameter of 50 mm, is being pulled by forces, the line of action of which is parallel to the axis of the bar but does not coincide with it. Determine the maximum eccentricity of the forces for which the maximum stress will not exceed by more than 25% the stress at the axis of the bar.

Answer: 3·4 mm.

6.45. Compressive forces were applied to two columns at point A (see figure). The compressive stress at point C was found to be the same in both columns. In which column, and by how much (as a percentage) is the stress at point B greater?

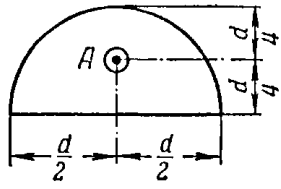
Answer: In the column of round cross-section the stress is 20% greater.



PROBLEM 6.45

6.46. Determine the diameter of a short wooden strut having a cross-section in the form of a semicircle. It is loaded by a longitudinal compressive force $P = 5{\cdot}6$ t applied to point A, the centre of the width and the thickness (see figure). $[\sigma] = 100$ kg/cm^2. Determine the required diameter of the strut if it works in simple compression.

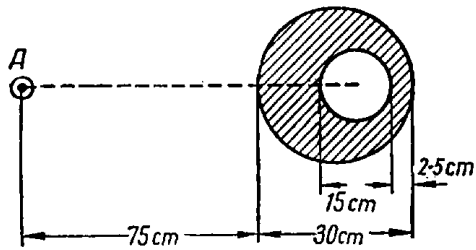
Answer: 15·2 cm; 12 cm.



PROBLEM 6.46

6.47. The cross-section of the vertical column of a radial drilling machine is shown in the figure. The thrust on the drill loads the column by a vertical tensile force passing through point A. Determine by how much (as a percentage) the maximum tensile stress in the given case is higher than the stress which would occur in the column of uniform wall thickness.

Answer: By 3·4%.

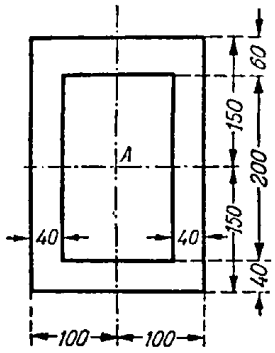


PROBLEM 6.47

6.48. When casting a hollow steel column of rectangular cross-section two parallel walls were erroneously made unequal in thickness (see figure).

Determine the value of the maximum and minimum stress in the column, caused by a longitudinal load of 400 t applied to point A. Determine also the value of the maximum normal stress in the column in the case where the thickness of the two walls in question is made the same (50 mm).

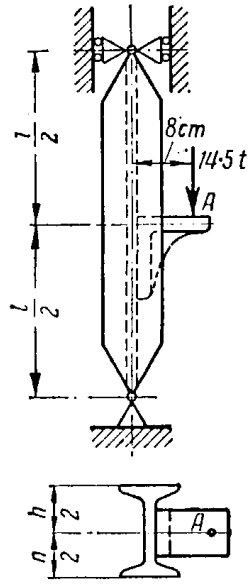
Answer: 1225 kg/cm²; 1007 kg/cm²; 1111 kg/cm².



PROBLEM 6.48

6.49. A load P of 14·5 t is transmitted through a bracket to a short I-profile column pinned at the ends (see figure). Select the size of the I-profile for an allowable stress of 1600 kg/cm².

Answer: I No. 22a.

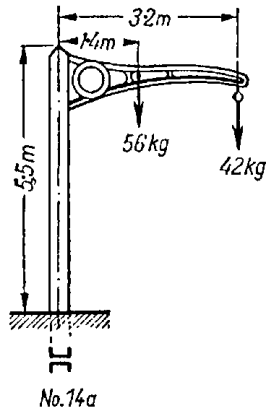


PROBLEM 6.49

6.50. A column serving as a support for a single tramway conductor consists of two No. 14a channels (see figure). The weight of the conductor is 42 kg, the weight of the bracket from which the

conductor is suspended is 56 kg. Taking into account the weight of the column itself, determine the maximum tensile and compressive stresses in the cross-section at the base of the column.

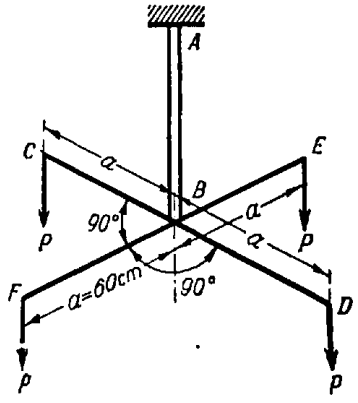
Answer: $+125$ kg/cm^2; -139 kg/cm^2.



PROBLEM 6.50

6.51. Four equal weights P (= 100 kg) are hung symmetrically from a crosspiece CDEF rigidly fastened to a steel tube AB (see figure). The tube AB has an external diameter of 60 mm, and a wall thickness of 2 mm. Determine the maximum tensile stress in the tube due to these four loads. Do the same for the case where one of the loads is taken off.

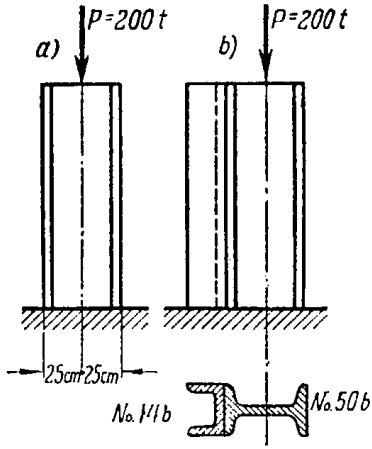
Answer: 110 kg/cm^2; 1255 kg/cm^2.



PROBLEM 6.51

6.52. A short No. 50b I-section stanchion loaded by a centrally applied compressive force $P = 200$ t (figure (a)), was reinforced by a No. 14b channel welded to the stanchion along its whole length (figure (b)). The force $P = 200$ t was applied to the reinforced column at the middle of the depth of the I-section. What was the maximum compressive stress in the original stanchion, and that in the stanchion reinforced by the channel?

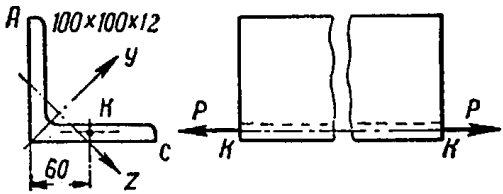
Answer: 1550 kg/cm²; 1684 kg/cm².



PROBLEM 6.52

6.53. A 100 mm × 100 mm × 12 mm angle is pulled by two forces $P = 10$ t applied to its ends at the mid-thickness of one of the legs, at a distance of 60 mm from the heel of the angle (see figure). Determine the position of the neutral axis, and the stress at points A, B and C of the cross-section of the angle.

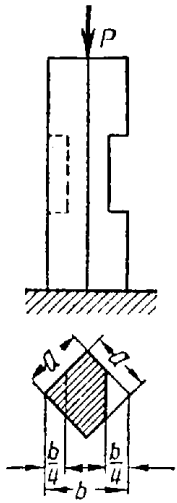
Answer: $a_z = -3{\cdot}7$ cm; $a_y = -6{\cdot}95$ cm; $\sigma_A = -191$ kg/cm²; $\sigma_B = +179$ kg/cm²; $\sigma_C = +1441$ kg/cm².



PROBLEM 6.53

6.54. What is the percentage increase in the maximum compressive stress in a short column of square cross-section (of side a), which is compressed by a centrally applied force P, if (a) a cut is made in it as indicated in the figure, (b) two such identical cuts are made in the opposite sides of the column?

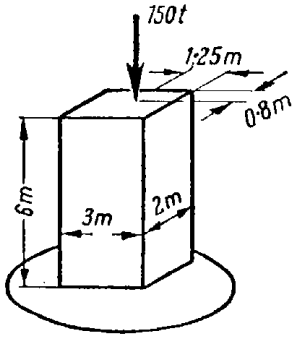
Answer: By 70·2%; by 33·3%.



PROBLEM 6.54

6.55. A wooden pillar, with a specific weight of 2 t/m³, is loaded as shown in the figure. Determine the maximum and minimum compressive stresses in its base, and indicate the points where they occur.

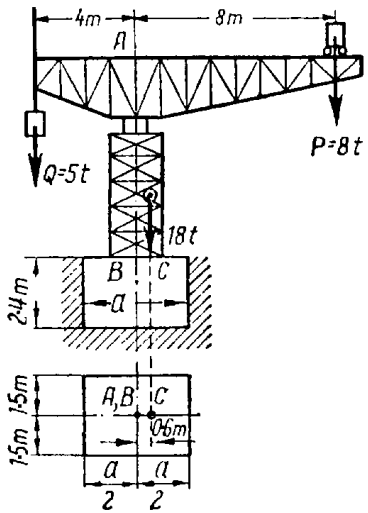
Answer: 6·45 kg/cm²; 0·95 kg/cm².



PROBLEM 6.55

6.56. A crane for lifting and displacing loads up to 8 t (see figure) is supported on a concrete foundation, with the axis AB of the crane passing through the centroid of the foundation. Assuming that the weight of the crane, equal to 18 t (without the weight of the load P and that of the counterbalance Q), is applied to the foundation at point C, 0·6 m from the axis AB, determine the dimension a of the side of the foundation for no tensile stresses to occur at its base. Determine what will be the maximum ground pressure for the chosen value of a. The specific gravity of the concrete is 2·2.

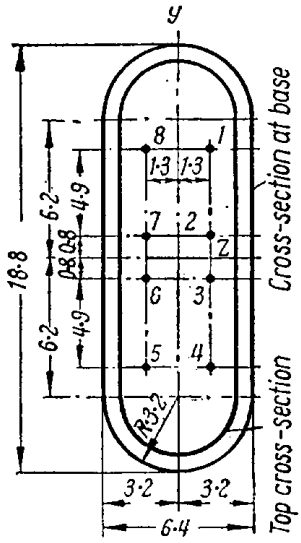
Answer: 3·68 ≈ 3·7 m; 16·1 t/m².



PROBLEM 6.56

6.57. In the figure the horizontal cross-section of a pier of a double line railway bridge is shown. The weight of the pier is 3200 t. Forces equal to 120 t each, due to the weight of the span structures of the bridge, are transmitted to the pier at points 1, 2, 3, 4, 5, 6, 7, and 8. In addition, forces equal to 210 t each, due to the weight of a train passing along one of the spans, are transmitted to the pier at points 1 and 2. Determine the position of the neutral axis, and the maximum and minimum compressive stresses in the horizontal cross-section of the pier at its base.

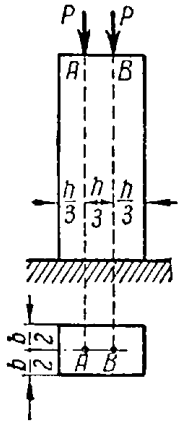
Answer: $a_y = -86{\cdot}7$ m; $a_z = -26{\cdot}6$ m; $\sigma_{max} = 49{\cdot}2$ t/m^2; $\sigma_{min} = 33{\cdot}0$ t/m^2.



PROBLEM 6.57

6.58. Equal forces are applied to points A and B of a column of rectangular cross-section (see figure). How will the maximum compressive stress in the material of the column change if one of the forces is removed?

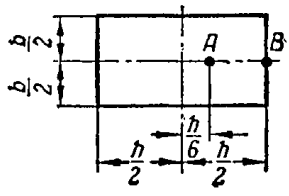
Answer: It remains unchanged.



PROBLEM 6.58

6.59. Due to what kind of stress (tensile or compressive) does the failure of a concrete column of rectangular cross-section take place (see figure), if a compressive force is applied to it: (a) at point A, (b) at point B? The ultimate strength of the concrete in compression is six times greater than its ultimate strength in tension.

Answer: (a) Compressive; (b) tensile.



PROBLEM 6.59

6.60. A bar having a $12 \times 20\ cm^2$ cross-section and a length of 120 cm, clamped at one end, is pulled by a longitudinal force $P_1 = 6\ t$, and bent by a transverse force $P_2 = 400\ kg$ (see figure).

Determine the normal stress at the corners of the rectangle in the critical cross-section of the bar.

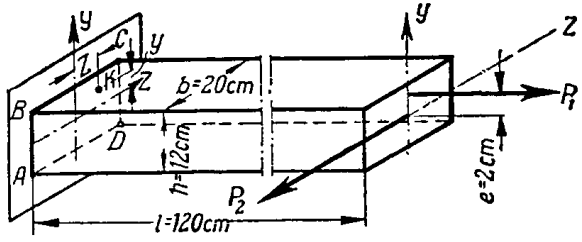
Answer:

$\sigma_A = -60\ kg/cm^2$ (compression);

$\sigma_B = -10\ kg/cm^2$ (compression);

$\sigma_C = +110\ kg/cm^2$ (tension);

$\sigma_D = +60\ kg/cm^2$ (tension).



PROBLEM 6.60

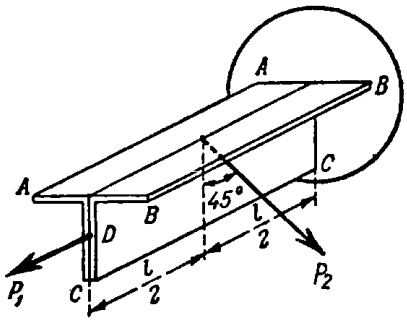
6.61. A cantilever, with a length $l = 2\ m$, is made up of two 120 mm $\times$ 80 mm $\times$ 10 mm angles rigidly joined to each other (see figure). A force $P_1 = 15\ t$, parallel to the axis of the beam,

is applied to its free end at point D (the mid-point of the depth of the angle); a force $P_2 = 800$ kg, acting in a plane perpendicular to the axis of the beam, passes through the shear centre at the mid-length, forming an angle of 45° with the principal centroidal axis of the cross-section.

Determine the value of the normal stress at points A, B and C of the critical section, and find the vertical and horizontal displacements of the centroid of the free end of the beam.

Answer:

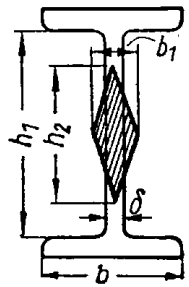
$$\sigma_A = +1841 \text{ kg/cm}^2;$$
$$\sigma_B = -601 \text{ kg/cm}^2;$$
$$\sigma_C = +22 \text{ kg/cm}^2;$$
$$f_{vert} = 0{\cdot}134 \text{ cm};$$
$$f_{hor} = 0{\cdot}662 \text{ cm}.$$



PROBLEM 6.61

6.62. How will the dimensions of the core of the cross-section of an I-profile (see figure) vary if its web thickness is reduced? What limiting dimensions will it have, if δ tends to zero?

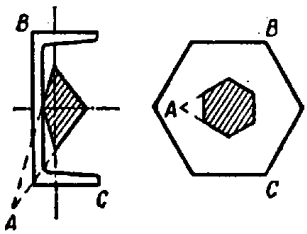
Answer: $b_{core} = \frac{b}{3}$; $b_{core} \approx h_1$.



PROBLEM 6.62

6.63. The cross-section of a channel and that of a regular hexagon are shown in the figure together with their respective cores. What will be the position of the neutral axis in each of these cross-sections, if a force, perpendicular to the plane of the cross-section, is applied at point A?

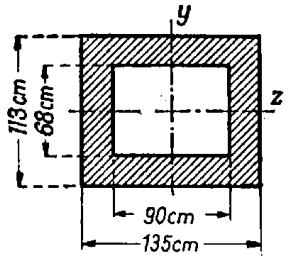
Answer: It passes through points B and C.



PROBLEM 6.63

6.64. A brickwork pillar has a hollow rectangular cross-section. The dimensions of the outer and the inner rectangles are 135 × 113 cm² and 90 × 68 cm² respectively (see figure). Determine the maximum distance from the centroid of the cross-section to the point of application of a compressive force for which there will be no tensile stresses in the pillar.

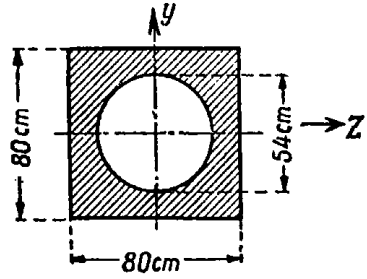
Answer: 30·9 cm along the z axis, and 26·9 cm along the y axis.



PROBLEM 6.64

6.65. The base cross-section of a factory chimney is shown in the figure. Determine the co-ordinates of the corner points of the core of the cross-section. Compare these with the co-ordinates of the corner points of the core of the 80 × 80 cm² square cross-section.

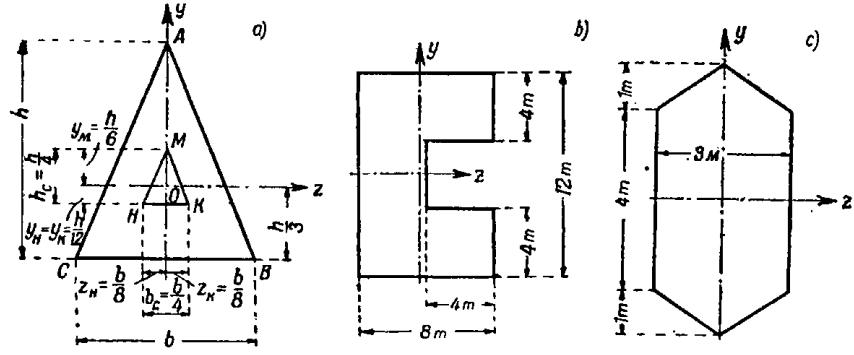
Answer: Chimney: $y_c = 0, z_c = \pm 18{\cdot}2$ cm; $z_c = 0$, $y_c = \pm 18{\cdot}2$ cm; square: $y_c = 0$, $z_c = \pm 13{\cdot}3$ cm; $z_c = 0$, $y_c = \pm 13{\cdot}3$ cm.



PROBLEM 6.65

6.66. Draw the cores for the cross-sections shown in figures (a), (b), and (c).

Answer: (a) See figure; (b) $y_c = 0$, $z_c = 1{\cdot}18$ m; $y_c = 0$, $z_c = 1{\cdot}44$ m; $z_c = 0$, $y_c = \pm 2{\cdot}36$ m; (c) $y_c = 0$, $z_c = \pm 0{\cdot}45$ m; $y_c = \pm 0{\cdot}722$ m, $z_c = \pm 0{\cdot}15$ m.

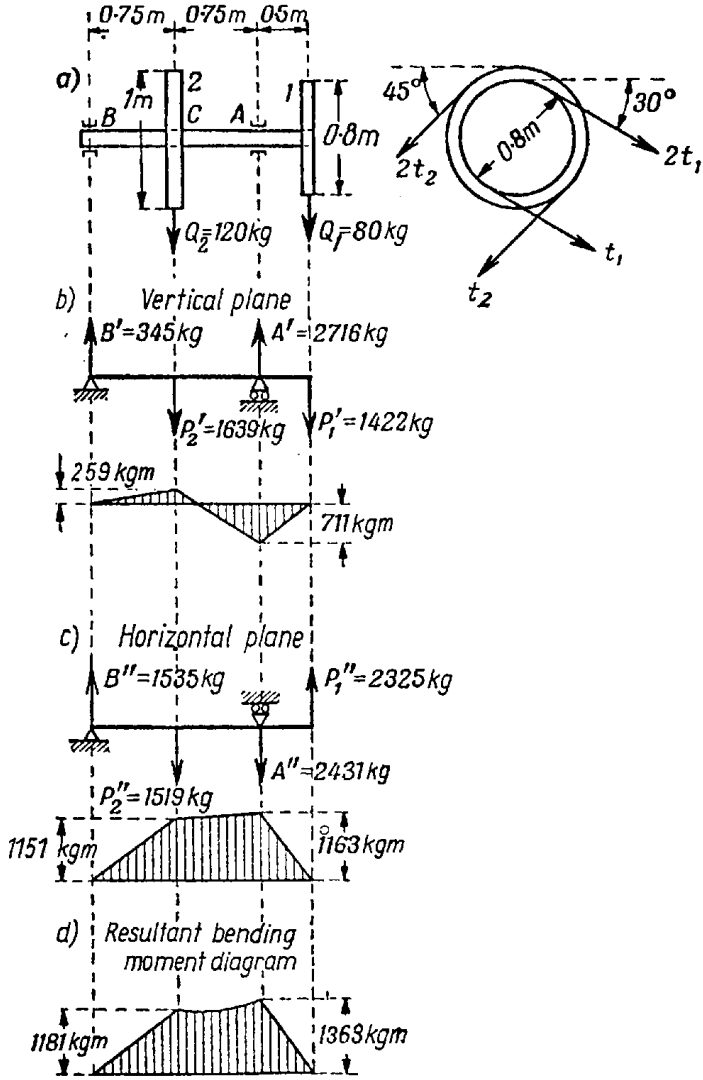


PROBLEM 6.66

27. COMBINED FLEXURE AND TORSION

6.67. Two pulleys (see figure) are mounted on a shaft; the first is 80 cm in diameter and weighs 80 kg, while the second is 1 m in diameter and weighs 120 kg. The pulleys are used as part of two belt drives. The belts, having their sides parallel, have angles of inclination of 30° and 45° with the horizontal, for first and second

pulley respectively. The belt from the first pulley goes to an electric motor; in this belt the force in the tight side is twice that in the slack side. The belt from the second pulley goes to a machine tool and again the force in the tight side is twice that in the slack side. The machine tool has a rating of 100 h.p.; the shaft rotates at



PROBLEM 6.67

200 rev/min. Applying the 4th theory of strength, determine the required diameter of the shaft, if the allowable stress is 800 kg/cm².

Solution. The shaft is subjected to flexure; in addition the section of it between the pulleys is also subjected to torsion. The torque is determined by the formula

$$M_t = \frac{2250}{\pi n} N = \frac{2250 \times 100}{3{\cdot}14 \times 200} = 358 \text{ kgm}.$$

We shall denote the tension in the slack side of the belt on the first pulley by t_1; then the tension in the tight side is $2t_1$. For the torque turning the pulley, which is equal to the torque M_t, we have the formula

$$M_1 = M_t = \frac{D_1 t_1}{2},$$

where D_1 is the diameter of the first pulley. Hence

$$t_1 = \frac{2M_t}{D_1} = \frac{2 \times 35{,}800}{80} = 895 \text{ kg}.$$

Similarly, for the second pulley, we have

$$t_2 = \frac{2M_t}{D_2} = \frac{2 \times 35{,}800}{100} = 716 \text{ kg}.$$

Thus at the sections where the pulleys are mounted the shaft is loaded by the inclined loads

$$R_1 = 3t_1 = 3 \times 895 = 2685 \text{ kg};$$

$$R_2 = 3t_2 = 3 \times 716 = 2148 \text{ kg}.$$

Further, the same sections are subjected to vertical forces due to the weight of the pulleys.

To determine the maximum bending moment we shall at first find the bending moments in the vertical and horizontal planes. To do this we shall resolve the forces R_1 and R_2 into the vertical and horizontal components; the weights of the pulleys will be

added to the vertical components. The vertical load due to the first pulley is

$$P'_z = Q'_1 + R_1 \sin 30° = 80 + (2685 \times 0{\cdot}5) = 1422 \text{ kg},$$

while that due to the second pulley is

$$P'_2 = Q_2 + R_2 \sin 45° = 120 + (2148 \times 0{\cdot}707) = 1639 \text{ kg}.$$

The horizontal load due to the first pulley is

$$P''_1 = R_1 \cos 30° = 2685 \times 0{\cdot}866 = 2325 \text{ kg},$$

while that due to the second pulley is

$$P''_2 = R_1 \cos 45° = 2148 \times 0{\cdot}707 = 1519 \text{ kg}.$$

The loads P''_1 and P''_2 will act in opposite directions (see figure (c)).

The reactions at the supports, due to the loads P'_1 and P'_2 acting in the vertical plane, are determined in the usual way: $A' = 2716$ kg and $B' = 345$ kg. We draw a diagram for the bending moment M'_b (see figure (b)).

The reactions at the supports, due to the loads P''_1 and P''_2 acting in the horizontal plane, are determined similarly: $A'' = 2341$ kg and $B'' = 1535$ kg. We draw a diagram for the bending moment M''_b (see figure (c)).

The resulting bending moment may reach the maximum value either at support A, or at section C, since both the bending moment diagrams are represented by broken straight lines, and have peaks at these points. The values of the resulting bending moment at these points are

$$M_A = \sqrt{(M'_A)^2 + (M''_A)^2} = \sqrt{711^2 + 1163^2} = 1363 \text{ kgm}$$

and

$$M_C = \sqrt{(M'_C)^2 + (M''_C)^2} = \sqrt{259^2 + 1151^2} = 1180 \text{ kgm}.$$

Thus the critical section is at A. The diagram of the resulting bending moment is represented in figure (d).

The effective bending moment, in accordance with the fourth theory of strength, is

$$M_e = \sqrt{M_A^2 + 0{\cdot}75 M_t^2} = \sqrt{1363^2 + (0{\cdot}75 \times 358^2)} = 1398 \text{ kgm}.$$

The required diameter of the shaft is found by the formula

$$d = \sqrt[3]{\frac{32M_e}{\pi[\sigma]}} = \sqrt[3]{\frac{32 \times 139{,}800}{3{\cdot}14 \times 800}} = 12{\cdot}1 \text{ cm}.$$

6.68. A solid steel shaft of circular cross-section is loaded at the critical section by a bending moment of 1·25 tm and a torque of 1·25 tm. The allowable stress $[\sigma]$ is 800 kg/cm². Determine the required diameter of the shaft, proceeding from the conditions of limiting strength in accordance with the third and fourth theories of strength.

Answer: 131 mm; 128 mm.

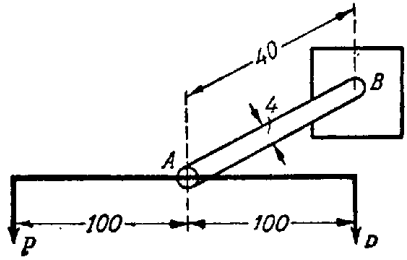
6.69. A hollow steel shaft is being twisted by a torque of 27 tm and bent by a moment of 14 tm. The internal diameter of the shaft is to be a half of the external diameter.

Applying the third theory of strength, determine the external and internal diameters of the shaft, if the allowable stress $[\sigma]$ is 1500 kg/cm².

Answer: 280 mm and 140 mm.

6.70. Determine the maximum effective stress in a circular steel rod AB, 40 cm in length and 4 cm in diameter, loaded by two equal loads $P = 100$ kg (see figure). What will be the effective stress in the rod, if one of the loads is removed? Apply the third theory of strength.

Answer: 1271 kg/cm²; 1518 kg/cm².



PROBLEM 6.70

6.71. A hollow cast iron shaft, with external and internal diameters of 200 mm and 120 mm respectively, transmits 300 h.p.

at 120 rev/min. The distance between the supporting bearings is 1·6 m. At the mid-point of this distance a concentrated load of 3 t is applied to the shaft.

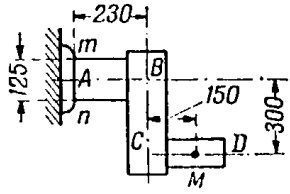
Determine the value of the maximum effective stress in the critical cross-section of the shaft, using the theory of maximum principal strain. $\mu = 0{\cdot}25$.

Answer: 263 kg/cm².

6.72. A cranked steel bar ABCD is fastened as indicated in the figure. The portion AB has a circular cross-section 125 mm in diameter. A force of 2 t is applied to point M perpendicularly to the plane of the drawing. Neglecting the shear stresses due to the shearing force, determine the principal stresses at the most critical point of section mn, and calculate the effective stresses according to the second, third and fourth theories of strength,

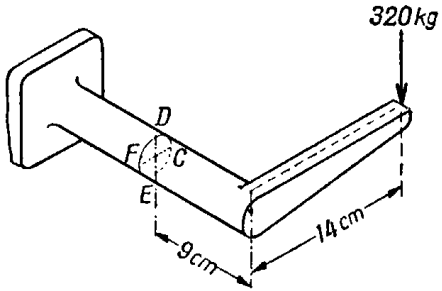
Do not take stress concentrations into consideration.

Answer: +451 kg/cm²; −54 kg/cm²; 467 kg/cm²; 505 kg/cm²; 480 kg/cm².



PROBLEM 6.72

6.73. A pawl (see figure) is loaded by a force of 320 kg. The diameter of the circular portion at section DEFC is 50 mm. Determine



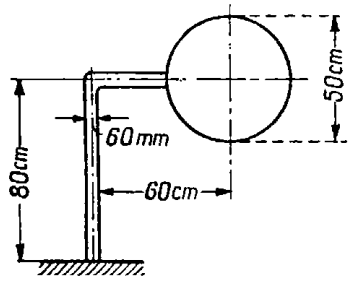
PROBLEM 6.73

the principal stress and the maximum shear stress at the point D. Proceeding from the fourth theory of strength, calculate the maximum effective stress in section DEFC.

Answer: $+335$ kg/cm^2; -90 kg/cm^2; 217 kg/cm^2; 394 kg/cm^2.

6.74. A round road sign is fastened to a hollow round post with an external diameter of 60 mm (see figure). Proceeding from the condition of limiting strength in accordance with the theory of maximum shear stress, determine the wall thickness of the post, if the allowable stress $[\sigma]$ is 600 kg/cm^2, and the maximum wind load on the sign is 200 kg/m^2.

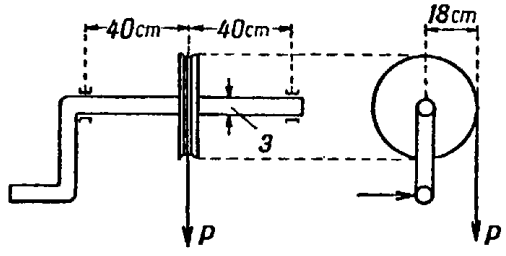
Answer: 2·65 mm.



PROBLEM 6.74

6.75. Proceeding from the condition of limiting strength in accordance with the theory of maximum shear stress, determine the maximum allowable value of a load P which may be raised by the winch (see figure). The shaft of the winch is of circular cross-section 30 mm in diameter. The allowable stress for the material of the shaft is 800 kg/cm^2.

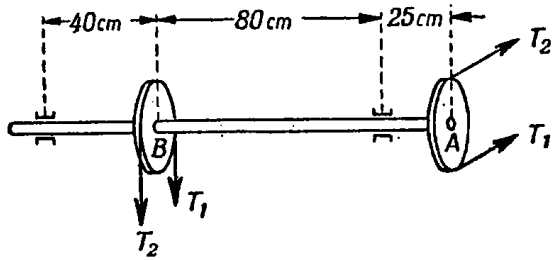
Answer: 79 kg.



PROBLEM 6.75

6.76. Two identical pulleys A and B are mounted on a shaft (see figure). The driving pulley A transmits 10 h.p. at 100 rev/min. Both sides of the belt on pulley A are horizontal, while on pulley B they are vertical. The tension T of the belt is 150 kg. The diameter of the pulleys is 60 cm. The weight of the pulleys may be neglected. Proceeding from the condition of limiting strength according to the theory of maximum shear stress, determine the required diameter of the shaft, if the allowable stress $[\sigma]$ is 800 kg/cm^2.

Answer: 59·6 mm $\approx$ 60 mm.

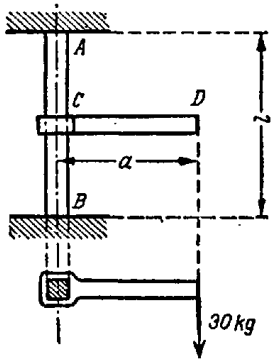


PROBLEM 6.76

6.77. Both ends of a steel bar AB, having a square cross-section of 20 $\times$ 20 mm^2, are clamped. At the middle of the span l (= 160 cm) a crosspiece CD, having the same cross-section and a length, a, of 30 cm, is rigidly joined to the bar. A load P (= 30 kg) is applied to the free end of the crosspiece (see figure).

Determine the effective stresses in bar AB and crosspiece CD by the theory of maximum shear stress, and find the displacement of point D in the direction of the load P.

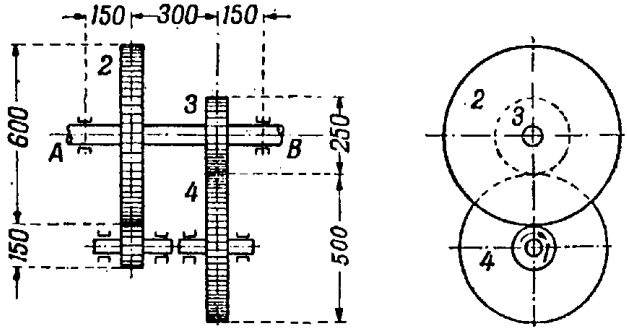
Answer: 704 kg/cm^2; 675 kg/cm^2; 9·4 mm.



PROBLEM 6.77

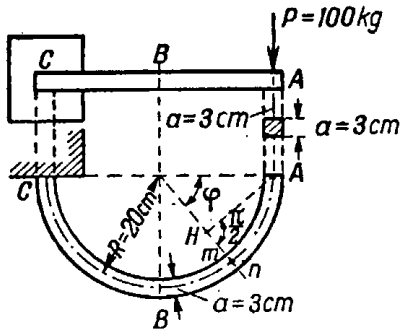
6.78. Determine the diameter of shaft AB of a spur gear drive (see figure). The torque on gear wheel 1 is 30 kgm. Apply the third and fourth theories of strength, neglecting the weight of the gear wheels. The allowable stress $[\sigma]$ is 800 kg/cm^2.

Answer: 57·8 mm; 56·2 mm.



PROBLEM 6.78

6.79. A steel rod ABC of a 3 × 3 cm^2 square cross-section, bent in the shape of a semicircle, is clamped at end C in a wall and loaded at the free end by a force $P = 100$ kg (see figure). Determine the maximum effective stresses in sections C and D of the rod, proceeding from the theory of maximum shear stress. Determine the vertical displacement of point A.



PROBLEM 6.79

Solution. The force P causes flexure and torsion of rod ABC. In an arbitrary section mn inclined at an angle to section A (see figure) the bending moment is

$$M_b = P \times HA = PR \sin \Phi,$$

whereas the torque is

$$M_t = P \times HK = PR(1 - \cos \Phi).$$

At section B, with $\Phi = \pi/2$, we have $M_b = M_t = PR$. The maximum normal stress due to flexure at this section is

$$\sigma_{max} = \frac{M_b}{W_b} = \frac{6PR}{a^3} = \frac{6 \times 100 \times 20}{3^3} = 444 \text{ kg/cm}^2.$$

The maximum shear stress due to torsion at the same section is

$$\tau_{max} = \frac{M_t}{W_t} = \frac{PR}{\beta a^3}.$$

For a square cross-section the coefficient β is 0·208 (see Table 17* of the book *Strength of Materials* by N. M. Belyayev, 1954 Edition); thus

$$\tau_{max} = \frac{100 \times 20}{0{\cdot}208 \times 3^3} = 356 \text{ kg/cm}^2.$$

The maximum effective stress, according to the third theory of strength, in section B is

$$\max \sigma_{eff} = \sqrt{\sigma_{max}^2 + 4\tau_{max}^2} = \sqrt{444^2 + (4 \times 356^2)} = 839 \text{ kg/cm}^2.$$

At section C, with $\Phi = \pi$, we have $M_b = 0$ and $M_t = 2PR$; therefore $\sigma = 0$, while $\tau_{max} = 2 \times 356 = 712$ kg/cm². The effective stress at this section is

$$\max \sigma_{eff} = \sqrt{4\tau_{max}^2} = 2\tau_{max} = 2 \times 712 = 1424 \text{ kg/cm}^2.$$

Thus section C is under worse conditions than section B.

The displacement of end A of rod ABC in the direction of the force P will be determined by means of Castigliano's theorem. Thereby the strain energy of flexure and torsion of the rod is taken

* Table 17 appears in Appendix B, Section 6.

into account. In an element of length $\mathrm{d}s = R\,\mathrm{d}\Phi$ the strain energy of flexure and torsion is

$$\mathrm{d}U_\mathrm{b} + \mathrm{d}U_\mathrm{t} = \frac{M_\mathrm{b}^2 R\,\mathrm{d}\Phi}{2EJ_\mathrm{b}} + \frac{M_\mathrm{t}^2 R\,\mathrm{d}\Phi}{2GJ_\mathrm{t}}.$$

The strain energy of the whole rod is

$$U = U_\mathrm{b} + U_\mathrm{t} = \int_0^\pi \left(\frac{M_\mathrm{b}^2 R}{2EJ_\mathrm{b}} + \frac{M_\mathrm{t}^2 R}{2GJ_\mathrm{t}}\right)\mathrm{d}\Phi.$$

According to Castigliano's theorem

$$f_\mathrm{A} = \frac{\partial U}{\partial P} = \int_0^\pi \left(\frac{M_\mathrm{b}\dfrac{\partial M_\mathrm{b}}{\partial P}R}{EJ_\mathrm{b}} + \frac{M_\mathrm{t}^2 R}{2GJ_\mathrm{t}}\right)\mathrm{d}\Phi.$$

Since $\partial M_\mathrm{b}/\partial P = R\sin\Phi$, $\partial M_\mathrm{t}/\partial P = R(1-\cos\Phi)$, $G = 8\times 10^5\ \mathrm{kg/cm^2} = 0{\cdot}4\,E$, $J_\mathrm{b} = a^4/12$ and $J_\mathrm{t} = \alpha a^4$, where $\alpha = 0{\cdot}14$ for a square cross-section (see the above-mentioned Table 17), we obtain

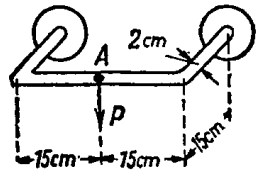
$$f_\mathrm{A} = \frac{PR^3}{Ea^4}\int_0^\pi \left[12\sin^2\Phi + \frac{(1-\cos\Phi)^2}{0{\cdot}4\alpha}\right]\mathrm{d}\Phi$$

$$= \frac{3\pi PR^3}{a^4E}\left(2 + \frac{E}{2G\alpha}\right) = \frac{3\times 3{\cdot}14\times 100\times 20^3}{3^4\times 2\times 10^6}$$

$$\times\left(2 + \frac{2\times 10^6}{2\times 8\times 10^5\times 0{\cdot}14}\right) = 0{\cdot}51\ \mathrm{cm}.$$

6.80. A rectangular steel bracket of circular cross-section 2 cm in diameter is fixed at both ends in a wall, and loaded by a force P (see figure). Proceeding from the condition of limiting strength depending on shear stress, determine the maximum allowable value of the force P, if the maximum allowable stress $[\sigma]$ is 800 kg/cm^2. Determine also the vertical displacement of point A.

Answer: 81·8 kg; 0·49 mm.

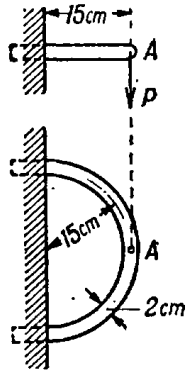


PROBLEM 6.80

6.81. A steel rod of circular cross-section 2 cm in diameter, bent in the shape of a semicircle is built into a wall at its ends. Proceeding from the condition of limiting strength in accordance with the theory of shear stress, determine, for an allowable stress $[\sigma]$ of 800 kg/cm^2, the maximum allowable value of the force P which can be applied to point A. Determine also the vertical displacement of point A.

Compare the result of the solution of the problem with the results of the solution of Problem 6.80.

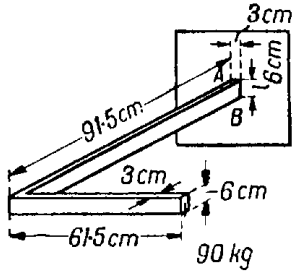
Answer: 70·7 kg; 0·39 mm.



PROBLEM 6.81

6.82. A cranked rod, having a rectangular cross-section of 3×6 cm^2, is clamped at one end and loaded by a concentrated force of 90 kg at the free end (see figure). Determine the effective stresses according to the third and fourth theories of strength at points A and B, the mid-points of the narrow and wide sides of the rectangle at the clamped section.

Answer: According to the third theory of strength: $\sigma_{\text{eff A}} = 787$ kg/cm^2; $\sigma_{\text{eff B}} = 812$ kg/cm^2; according to the 4th theory of strength: $\sigma_{\text{eff A}} = 718$ kg/cm^2; $\sigma_{\text{eff B}} = 703$ kg/cm^2.



PROBLEM 6.82

28. GENERAL CASE OF COMBINED LOADING

6.83. A hollow shaft with an external diameter D and an internal diameter $D/2$ is being twisted by a torque of 4·5 tm, and compressed by an axial force of 50 t. The allowable stress $[\sigma]$ is 1200 kg/cm^2. Determine the diameters of the shaft, proceeding from the fourth theory of strength.

Answer: 202 mm; 101 mm.

6.84. A steel tube, with an internal diameter of 100 mm and a wall thickness of 5 mm, is subjected to an internal pressure of 10 atm (gauge) not causing longitudinal stress, an axial compressive force of 8 t, and a torque of 500 kgm. Determine the magnitude of the principal stresses, and the angle of inclination (to the axis of the tube) of the plane having the principal tensile stress.

Answer: $+508$ kg/cm^2; -893 kg/cm^2; $32°41'$.

6.85. A shaft 22 cm in diameter, carrying a propeller screw at the end of a 45 cm long overhang, transmits 4000 h.p. at 300 rev/min. The tensile force on the screw is 17 t, its weight is 2 t. Determine the magnitude of the principal stresses in the most critical point on the surface of the shaft, and construct a diagram showing the variation of the principal stresses at this point during one revolution of the shaft. Determine the magnitude of the effective

stress at the same point in accordance with the theory of shear strain energy.

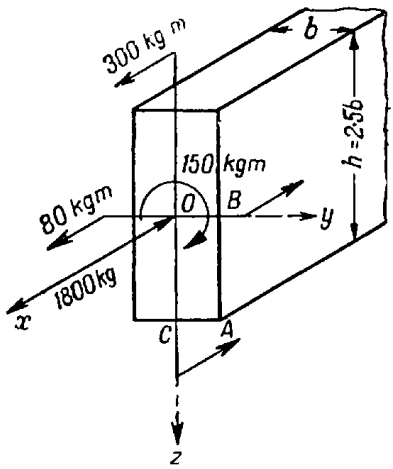
Answer: $+524$ kg/cm^2; -398 kg/cm^2; 801 kg/cm^2.

6.86. A bar having a rectangular cross-section of 10×3 cm^2 is twisted by a torque of 90 kgm, and subjected to tension by a centroidal force of 2·4 t.

Determine the value of the maximum effective stress by the third theory of strength.

Answer: 731 kg/cm^2.

6.87. Determine the dimensions of a crank web of rectangular cross-section with the ratio of sides $h : b = 2{\cdot}5$, loaded as indicated in the figure. Apply the theory of maximum shear stress. The allowable stress $[\sigma]$ is 800 kg/cm^2.



PROBLEM 6.87.

Solution. The most critical points in the cross-section are:

(1) point A at which the normal stress due to the force N and the moments M_y and M_z will be of the same sign (compression); at this point there will be no shear stress due to torsion;

(2) point B at which the shear stress due to torsion will be the greatest, and the compressive normal stress is caused only by the force N and the moment M_z;

(3) point C at which the shear stress due to torsion will be somewhat less than at point B; on the other hand the compressive normal stress due to the force N and the moment M_y is greater than that due to the moment M_z.

We shall formulate the conditions of limiting strength of the material for each of these points, proceeding from the theory of maximum shear stress.

The geometrical characteristics of the cross-section, expressed in terms of the dimension b of the web, will be

$$F = bh = 2{\cdot}5b^2;$$

$$W_y = \frac{bh^2}{6} = \frac{2{\cdot}5^2b^3}{6} = 1{\cdot}024b^3,$$

$$W_z = \frac{hb^3}{6} = \frac{2{\cdot}5b^3}{6} = 0{\cdot}417b^3$$

and

$$W_t = \beta b^3 = 0{\cdot}647b^3$$

(at $h : b = 2{\cdot}5$ the coefficient $\beta = 0{\cdot}647$).

The condition of limiting strength of the material at A, the point furthermost from the y and z axes, has the form

$$\sigma_{\text{eff A}} = |\sigma_A| = \frac{N}{F} + \frac{M_y}{W_y} + \frac{M_z}{W_z} = \frac{1800}{2{\cdot}5b^2} + \frac{30{,}000}{1{\cdot}024b^3} + \frac{8000}{0{\cdot}417b^3}$$

$$= \frac{720}{b^2} + \frac{28{,}800}{b^3} + \frac{19{,}200}{b^3} = \frac{720}{b^2} + \frac{48{,}000}{b^3}$$

$$\leq [\sigma] = 800 \text{ kg/cm}^2.$$

Assuming that $\sigma_{\text{eff A}} = [\sigma]$ we obtain the equation for b:

$$800b^3 - 720b - 48{,}000 = 0 \quad \text{or} \quad b^3 - 0{\cdot}9b - 60 = 0.$$

By trial and error we find that $b = 3{\cdot}99$ cm.

The normal stress at point B is

$$|\sigma_B| = \frac{N}{F} + \frac{M_z}{W_z} = \frac{720}{b^2} + \frac{19{,}200}{b^3},$$

while the shear stress is

$$\tau_B = \frac{M_t}{W_t} = \frac{15{,}000}{0{\cdot}647b^3} = \frac{23{,}180}{b^3}.$$

The condition of limiting strength has the form

$$\sigma_{\text{eff B}} = \sqrt{\sigma_B^2 + 4\tau_B^2} = \sqrt{\left(\frac{720}{b^2} + \frac{19{,}200}{b^3}\right)^2 + 4\left(\frac{23{,}180}{b^3}\right)^2}$$

$$\leq [\sigma] = 800 \text{ kg/cm}^2.$$

Assuming that $\sigma_{\text{eff B}} = [\sigma]$, we obtain the equation

$$b^6 - 0{\cdot}18b^2 - 43{\cdot}2b - 3934 = 0,$$

whence by trial and error we find that $b = 4{\cdot}00$ cm.

The normal stress at point C is

$$|\sigma_C| = \frac{N}{F} + \frac{M_y}{W_y} = \frac{720}{b^2} + \frac{28{,}800}{b^3},$$

while the shear stress is

$$\tau_C = \gamma \frac{M_t}{W_t} = 0{\cdot}774 \times \frac{15{,}000}{0{\cdot}647b} = \frac{17{,}950}{b^3}$$

(at $h : b = 2{\cdot}5$ the coefficient $\gamma = 0{\cdot}774$).

The condition of limiting strength has the form

$$\sigma_{\text{eff. C}} = \sqrt{\sigma_C^2 + 4\tau_C^2} = \sqrt{\left(\frac{720}{b^2} + \frac{28{,}800}{b^3}\right)^2 + 4\left(\frac{17{,}950}{b^3}\right)^2}$$

$$\leq [\sigma] = 800 \text{ kg/cm}^2.$$

Assuming that $\sigma_{\text{eff. C}} = [\sigma]$, we obtain the equation

$$b^6 - 0{\cdot}81\,b^2 - 64{\cdot}8b - 3310 = 0,$$

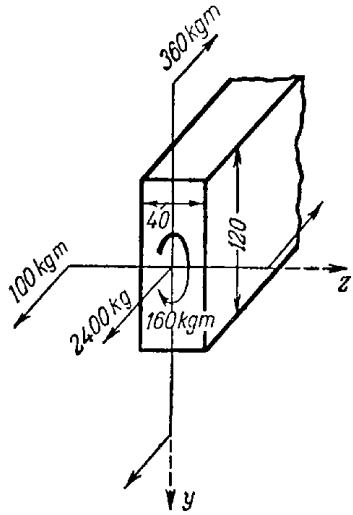
whence by trial and error we find that $b = 3{\cdot}91$ cm.

Thus the most critical point is B. The dimensions of the crank web must be taken as $b = 4$ cm and $h = 2{\cdot}5b = 10$ cm.

6.88. The web of a crankshaft, having a rectangular cross-section of 4×12 cm^2, is loaded as shown in the figure.

Determine the position of the neutral axis and, proceeding from the third theory of strength, calculate the effective stress in the three most critical points of the cross-section of the web (at the corner of the rectangle, and at the mid-points of its sides).

Answer: $a_y = -0{\cdot}8$ cm; $a_z = 0{\cdot}32$ cm; $\sigma_{\text{eff}} = 738$ kg/cm^2; $\sigma_{\text{eff}} = 722$ kg/cm^2; $\sigma_{\text{eff}} = 634$ kg/cm^2.

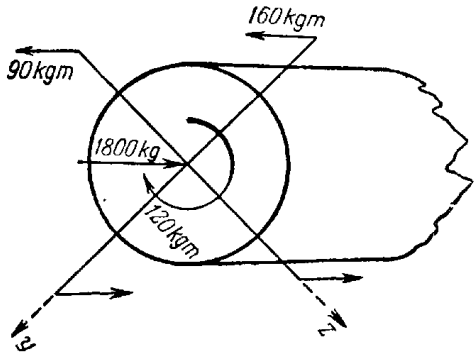


PROBLEM 6.88

6.89. The pin of a crankshaft is loaded as shown in the figure.

Proceeding from the condition of limiting strength in accordance with the third theory of strength, determine the diameter of the pin, if $[\sigma]$ is 1200 kg/cm^2. Give the co-ordinates of the critical point.

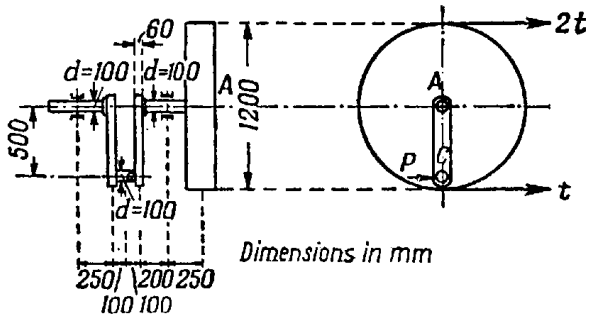
Answer: 58 mm; $z_0 = 1{\cdot}42$ cm; $y_0 = 2{\cdot}53$ cm.



PROBLEM 6.89

6.90. Check the strength of the pins and webs of a crankshaft (see figure). The pulley A transmits 100 h.p. at 150 rev/min. The allowable stress is 800 kg/cm². Apply the theory of maximum shear stress. Neglect the weight of the pulley. The width (AC) of the web is 130 mm.

Answer: The shaft: $\sigma_{\text{eff}} = 779$ kg/cm² $< [\sigma]$; the crankpin: $\sigma_{\text{eff}} = 585$ kg/cm² $< [\sigma]$; the right-hand web: $\sigma_{\text{eff}} = 743$ kg/cm² $< [\sigma]$; the left-hand web: $\sigma_{\text{eff}} = 428$ kg/cm² $< [\sigma]$.

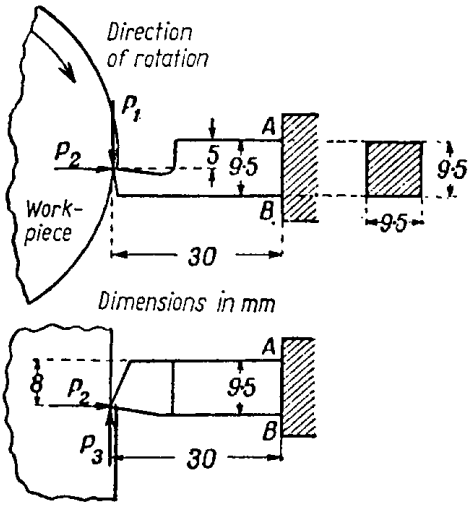


PROBLEM 6.90

6.91. The tip of a lathe cutting tool (see figure), whilst turning a component, is subjected to the action of three mutually perpendicular forces: P_1—the tangential force (the resistance to cutting in the narrow sense of the word), P_2—the radial force tending to push the tool back, and P_3—the axial, or feed force parallel to the axis of the component.

Proceeding from the theory of maximum shear stress, determine the effective stress at the three most critical points, and at the centre of the section AB of the tool (assuming the tool being clamped at this section), if $P_1 = 60$ kg, $P_2 = 0{\cdot}15\,P_1$ and $P_3 = 0{\cdot}35\,P_1$. In the calculation take into consideration also the shear stresses due to the direct shearing effect of the forces. Show what stresses will act on the faces of the element at the points stated above.

Answer: 1733 kg/cm²; 1305 kg/cm²; 619 kg/cm²; 212 kg/cm².



PROBLEM 6.91

6.92. Determine the magnitude of the three components of the displacement, and find the magnitude of the resultant displacement of end A of a cranked steel rod ABCD, 3 cm in diameter. The load P is 40 kg, the modulus E is 2×10^6 kg/cm², and the modulus G is 8×10^5 kg/cm² (see figure (a)).

Answer: The displacement of point A in the direction of the force $S = 1$ is

$$\delta_S = -\frac{P l_1 l_3^2}{2 E J_b} = -0{\cdot}189 \text{ cm};$$

point A is displaced vertically in the direction opposite to that of the force $S = 1$, i.e. upwards.

The displacement in the direction of the force $H = 1$ is

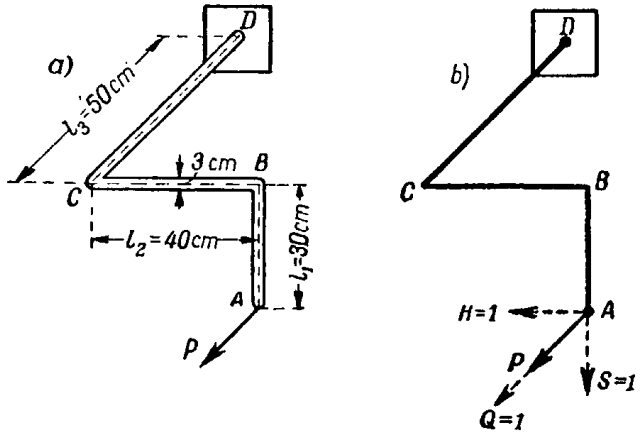
$$\delta_H = \frac{Pl_2 l_3^2}{2EJ_b} = 0{\cdot}252 \text{ cm}.$$

The displacement in the direction of the force $Q = 1$ is

$$\delta_Q = 1{\cdot}008 \text{ cm}.$$

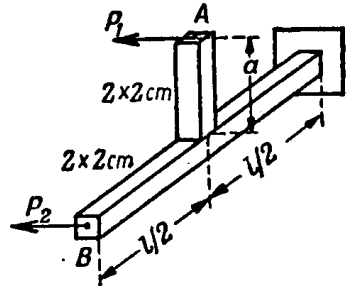
The resultant displacement of point A in space is

$$\delta = \sqrt{\delta_Q^2 + \delta_S^2 + \delta_H^2} = \sqrt{1{\cdot}008^2 + 0{\cdot}189^2 + 0{\cdot}252^2} = 1{\cdot}06 \text{ cm}.$$



PROBLEM 9.92

6.93. Determine the horizontal displacement of points A and B of a cranked steel bar (see figure) having a square cross-section



PROBLEM 6.93

of 2×2 cm^2 throughout. $P_1 = 6$ kg; $P_2 = 4$ kg; $a = 20$ cm; $l = 80$ cm.

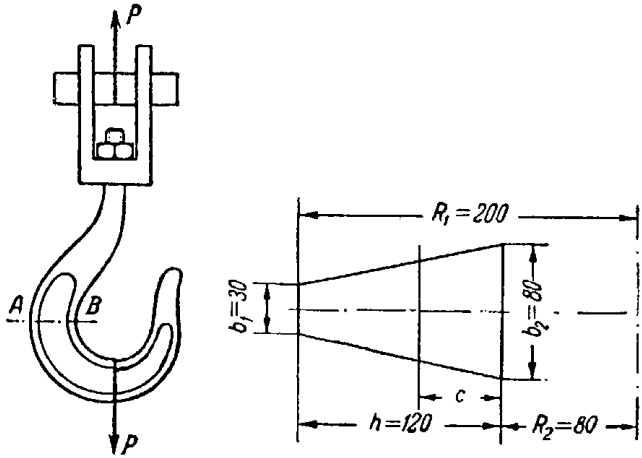
Answer: $\delta_A = 0{\cdot}188$ cm; $\delta_B = 0{\cdot}376$ cm.

29. CURVED BARS

6.94. Determine the maximum tensile and compressive stresses in the critical cross-section of a hook lifting a load $P = 10$ t. The dimensions of the hook are given in the figure. (The rounding-off of the corners of the cross-section is not shown, and in the calculation is not to be taken into account.)

Solution. Section AB will be the critical cross-section where the bending moment ($M = PR_0$) and the normal force ($N = P$) have their greatest values. The stresses in the inner and outer fibres of the cross-section are determined as

$$\sigma = \frac{N}{F} + \frac{M}{S}\frac{z}{\varrho}.$$



PROBLEM 6.94

The position of the centroid will be determined by its distance from the inner face of the cross-section

$$c = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} = \frac{12 \times 14}{3 \times 11} = 5{\cdot}09 \text{ cm}.$$

The position of the neutral layer will be determined by its radius r

$$r = \frac{h\,\dfrac{b_1 + b_2}{2}}{\left(b_1 + R_1\,\dfrac{b_2 - b_1}{h}\right)\log_e \dfrac{R_1}{R_2} - (b_2 - b_1)}$$

$$= \frac{12 \times 5{\cdot}5}{\left(3 + 20 \times \dfrac{5}{12}\right)\log_e \dfrac{20}{8} - 5} = \frac{66}{(11{\cdot}33 \times 2{\cdot}303 \times 0{\cdot}3979) - 5}$$

$$= \frac{66}{10{\cdot}4 - 5} = 12{\cdot}22 \text{ cm}.$$

The quantities necessary for the calculation of the stresses are

$$R_0 = R_2 + c = 8 + 5{\cdot}09 = 13{\cdot}09 \text{ cm};$$

$$z_0 = R_0 - r = 13{\cdot}09 - 12{\cdot}22 = 0{\cdot}87 \text{ cm};$$

$$S = F \times z_0 = 5{\cdot}5 \times 12 \times 0{\cdot}87 = 57{\cdot}4 \text{ cm};$$

$$z_1 = h - c + z_0 = 12 - 5{\cdot}09 + 0{\cdot}87 = 7{\cdot}78 \text{ cm};$$

$$z_2 = c - z_0 = 5{\cdot}09 - 0{\cdot}87 = 4{\cdot}22 \text{ cm}.$$

We determine the stresses

$$\sigma_2 = +\frac{10{,}000}{66} + \left(\frac{10{,}000 \times 13{\cdot}09}{57{\cdot}4} \times \frac{4{\cdot}22}{8}\right)$$

$$= 152 + 1200 = 1352 \text{ kg/cm}^2;$$

$$\sigma_1 = +\frac{10{,}000}{66} - \left(\frac{10{,}000 \times 13{\cdot}09}{57{\cdot}4} \times \frac{7{\cdot}78}{20}\right)$$

$$= 152 - 890 = -738 \text{ kg/cm}^2.$$

6.95. A hook has a trapezoidal cross-section with a base of 5 cm on the inner side, and a base of 2·5 cm on the outer side. The depth of the critical cross-section is 10 cm; the radius of the inner fibres is 10 cm. The line of action of the force passes through the centre of curvature of the hook.

What load can be lifted by the hook, if the allowable stress is 880 kg/cm²?

Answer: 2920 kg ($r = 13{\cdot}898$ cm).

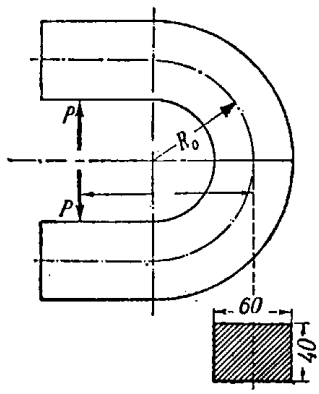
6.96. The radius of the inner fibres of a curved bar of trapezoidal cross-section is equal to the depth of the cross-section. The base of the trapezium on the concave side is four times greater than the base on the convex side. Find the ratio of the stresses in the extreme fibres of the curved bar to the stresses in the same fibres of a straight bar subjected to the same bending moment.

Answer: 1·28; 0·79 ($r = 1{\cdot}350h$).

6.97. A bar of rectangular cross-section, with a width of 6 cm and a thickness of 4 cm, is bent in the shape of a horseshoe having a mean radius R of 7 cm (see figure). Two equal and opposite forces of 1 t each are applied at a distance $a = 12$ cm from the centre line of the middle section so that they tend to straighten the rod.

Find the maximum tensile and maximum compressive stresses, and construct a diagram showing the variation of the normal stresses over the central section.

Answer: 750 kg/cm²; -340 kg/cm² ($r = 6{\cdot}548$ cm).



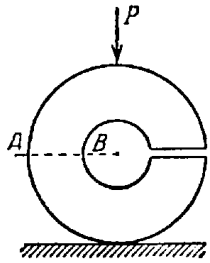
PROBLEM 6·97

6.98. Find the load-carrying capacity of a hook of rectangular cross-section. The thickness of the hook is 7·5 cm; the radius of the inner fibres is 15 cm, while that of the outer fibres is 25 cm. The line of action of the force passes at a distance of 7·5 cm from the inner fibres. The allowable stress is 700 kg/cm².

Answer: 5·2 t ($r = 19{\cdot}577$ cm).

6.99. The ring represented in the figure is made of a bar of circular cross-section 8 cm in diameter. The internal diameter of the ring is 12 cm. The ring is compressed by a force $P = 2$ t. Find the stress at points A and B.

Answer: 260 kg/cm^2; -607 kg/cm^2 ($r = 9{\cdot}581$ cm).



PROBLEM 6.99

6.100. A curved bar of circular cross-section 10 cm in diameter has a radius of curvature of 2·5 cm at the inner fibres. The bending moment acting on the bar causes a tensile stress of 200 kg/cm^2 in the inner fibres. What will be the stress in the outer fibres?

Answer: 59 kg/cm^2 ($r = 6{\cdot}545$ cm).

6.101. The radius of curvature of the inner fibres of a hollow curved bar is 1·25 cm. The cross-section is a hollow square having an outer side of 2·5 cm, and an inner side of 1·5 cm.

Find the ratio of the stress in the extreme fibres of the curved bar to the stresses in the same fibres of a straight bar subjected to the same bending moment.

Answer: 1·44; 0·78 ($r = 2{\cdot}200$ cm).

6.102. Determine the position of the neutral axis of a curved bar subjected to flexure by a couple. The cross-section of the bar is an I-profile having dimensions indicated in the figure. The numerical values of these dimensions are as follows:

$$b_3 = 3 \text{ cm}; \quad b_2 = 1 \text{ cm}; \quad b_1 = 2 \text{ cm};$$
$$R_4 = 3 \text{ cm};$$
$$h_1 = 1 \text{ cm}; \quad h_2 = 2 \text{ cm}; \quad h_3 = 1 \text{ cm}.$$

Solution. An accurate solution yields the following result

$$r = \frac{F}{\int_F \frac{\mathrm{d}F}{\rho}} = \frac{b_1h_1 + b_2h_2 + b_3h_3}{b_1 \log_e \frac{R_1}{R_2} + b_2 \log_e \frac{R_2}{R_3} + b_3 \log_e \frac{R_3}{R_4}}$$

$$= \frac{2 + 2 + 2}{2 \log_e \frac{7}{6} + \log_e \frac{3}{2} + 3 \log_e \frac{4}{3}} = \frac{7}{1{\cdot}577} = 4{\cdot}44 \text{ cm}.$$

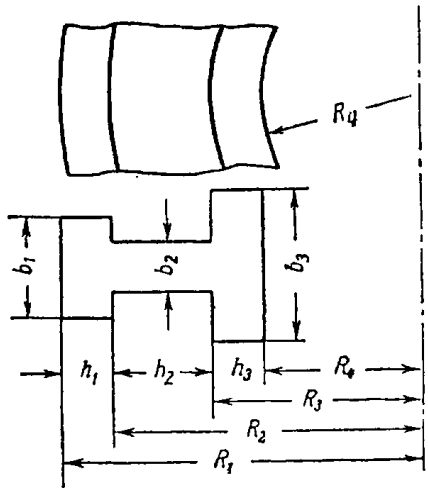
For a complicated cross-section the integration will become difficult. Therefore it will be replaced by the summation $\sum \Delta F/\varrho$, having divided the cross-section into small areas.

For an approximate solution of this problem we shall divide the whole cross-section into 11 strips parallel to the base of the I-profile each 4 mm high, except the third and the last strip, each of which, is 2 mm high. The calculation of the denominator of the formula or r is carried out in tabular form:

No.	ΔF, cm^2	ϱ, cm	$\frac{\Delta F}{\varrho}$, cm
1	1·2	3·2	0·375
2	1·2	3·6	0·333
3	0·6	3·9	0·154
4	0·4	4·2	0·0952
5	0·4	4·6	0·0870
6	0·4	5·0	0·0800
7	0·4	5·4	0·0742
8	0·4	5·8	0·0690
9	0·8	6·2	0·1290
10	0·8	6·6	0·1211
11	0·4	6·9	0·0580

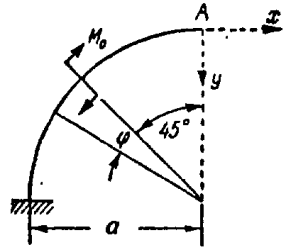
$$\sum \frac{\Delta F}{\varrho} = 1{\cdot}5755.$$

As shown in the table the results of the calculation are significantly close to the accurate values.



PROBLEM 6.102

6.103. Find the angle of rotation, and the vertical and horizontal displacements of section A of the curved bar shown in the figure. The radius of the axis of the bar is a.



PROBLEM 6.103

Solution. To determine the deformations of the curved bar we shall use Mohr's formula

$$\delta = \frac{1}{EJ}\int_S M(\Phi)M^0\,ds + \frac{1}{EF}\int_S N(\Phi)N^0\,ds.$$

Over the right-hand portion, from the point of application of M_0 to point A, $M(\Phi)$ and $N(\Phi)$ are zero; therefore in the problem

being considered it will be necessary to calculate the integrals only for the left-hand portion (from $\Phi = 0$ to $\Phi = \pi/4$):

$$\theta_A = \frac{1}{EJ}\int_0^{\frac{\pi}{4}} M_0 \times 1a\,d\Phi = \frac{\pi M a}{4EJ};$$

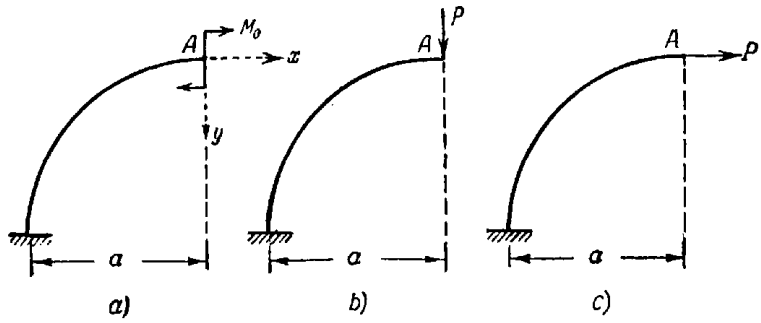
$$y_A = \frac{1}{EJ}\int_0^{\frac{\pi}{4}} M_0 a \sin\left(\frac{\pi}{4} + \Phi\right) a\,d\Phi = \frac{M_0 a^2 \cos\frac{\pi}{4}}{EJ}$$

$$= 0{\cdot}707\,\frac{M_0 a^2}{EJ};$$

$$x_A = \frac{1}{EJ}\int_0^{\frac{\pi}{4}} M_0 a\left[1 - \cos\left(\frac{\pi}{4} + \Phi\right)\right] a\,d\Phi$$

$$= \frac{M_0 a^2}{EJ}\left[\frac{\pi}{4} - 1 + \sin\frac{\pi}{4}\right] = 0{\cdot}492\,\frac{M_0 a^2}{EJ}.$$

6.104. Determine the angle of rotation θ_A, the vertical displacement y_A and the horizontal displacement x_A of the free end for each loading diagram of a curved bar (see figure) of uniform cross-section, bent along a circle of radius a.



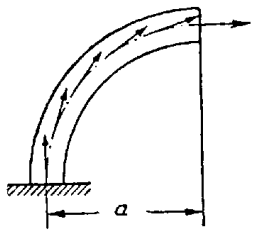
PROBLEM 6.104

Answer:

	(a)	(b)	(c)
θ_A	$\dfrac{\pi M_0 a}{2EJ}$	$\dfrac{Pa^2}{EJ}$	$0{\cdot}571\,\dfrac{Pa^2}{EJ}$
y_A	$\dfrac{M_0 a^2}{EJ}$	$\dfrac{\pi\,Pa^3}{4\,EJ}+\dfrac{\pi Pa}{4EF}$	$\dfrac{Pa^3}{2EJ}-\dfrac{Pa}{2EF}$
x_A	$0{\cdot}571\,\dfrac{M_0 a^2}{EJ}$	$\dfrac{Pa^3}{2EJ}-\dfrac{Pa}{2EF}$	$0{\cdot}356\,\dfrac{Pa^3}{EJ}+\dfrac{\pi}{4}\,\dfrac{Pa}{EF}$.

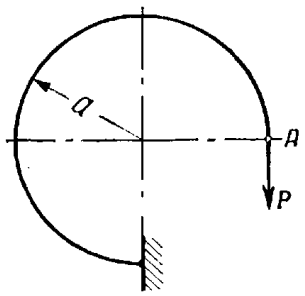
6.105. Find the vertical displacement of the free end of a curved bar the axis of which represents a quarter-circle of radius a. A load q is uniformly distributed along the axis of the rod and directed tangentially to it as shown in the figure. In determining the displacements take into consideration both the influence of the moments and that of the normal forces.

Answer: $\dfrac{qa^4}{EJ}\left[1-\dfrac{\pi}{4}\right]-\dfrac{\pi}{4}\,\dfrac{qa^2}{EF}$.



PROBLEM 6.105

6.106. Will point A of the curved bar represented in the figure be displaced in the horizontal direction under the action of the load P?



PROBLEM 6.106

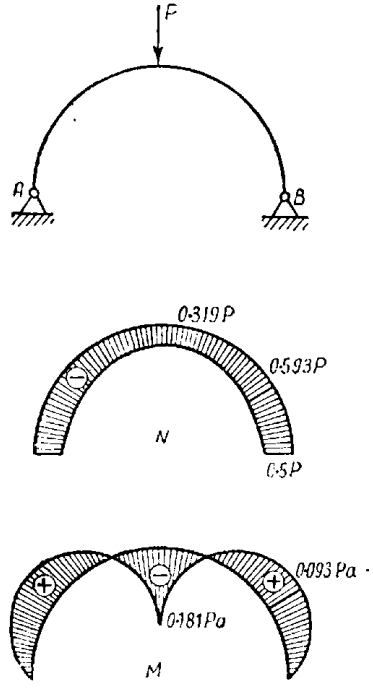
How can the direction of this displacement be derived without calculation? What will be the magnitude of the displacement?

Answer: It is displaced to the left. $x_A = \frac{Pa^3}{2EJ} - \frac{Pa}{2EF}$.

6.107. Determine the bending moment and the axial thrust (plot diagrams) for a circular arch, due to the action of a concentrated force applied to the apex.

Solution. This system is statically indeterminate. We choose the horizontal reaction H_B at hinge B for the unknown redundancy. The vertical components of the reactions at hinges A and B are determined from the equations of equilibrium. They are identical

$$V_A = V_B = \frac{P}{2}.$$



PROBLEM 6.107

For the determination of H_B ($H_A = H_B = H$) it is necessary to make use of the condition that in the direction of H there is no displacement at hinge B:

$$x_B = 0, \quad \text{or} \quad \int_S M(\Phi) \frac{\partial M(\Phi)}{\partial H} \, ds = 0.$$

Because of symmetry it is possible, for the integration, to consider only one half of the arch. Then

$$\int_0^{\frac{\pi}{2}} \left[-\frac{Pa}{2}(1 - \cos \Phi) + Ha \sin \Phi \right] a \sin \Phi \times a \, d\Phi = 0.$$

After integrating and solving for H we have

$$H = \frac{P}{\pi}.$$

It is now possible to construct the diagrams of $M(\Phi)$ and $N(\Phi)$, using the equations:

$$M(\Phi) = -\frac{P}{2} a(1 - \cos \Phi) + \frac{P}{\pi} a \sin \Phi$$

$$= Pa \left(\frac{\cos \Phi}{2} - \frac{1}{2} + \frac{\sin \Phi}{\pi} \right);$$

$$N(\Phi) = -\frac{P}{2} \cos \Phi - \frac{P}{\pi} \sin \Phi = -P \left(\frac{\cos \Phi}{2} + \frac{\sin \Phi}{\pi} \right).$$

The results of the calculation are given in the table:

Φ	0	30°	45°	60°	90°	32°30′
$\frac{M(\Phi)}{Pa}$	0	0·092	0·078	0·026	−0·181	0·093
$\frac{N(\Phi)}{P}$	−0·5	−0·592	−0·578	−0·526	−0·319	−0·593

We shall now determine the position of the section with the maximum value of $M(\Phi)$:

$$\frac{\mathrm{d}M(\Phi)}{\mathrm{d}\Phi} = 0;$$

$$-\frac{\sin\Phi}{2} + \frac{\cos\Phi}{\pi} = 0;$$

$$\frac{\sin\Phi}{\cos\Phi} = \frac{2}{\pi} = 0{\cdot}637; \quad \Phi = \tan^{-1} 0{\cdot}637 = 32°30'.$$

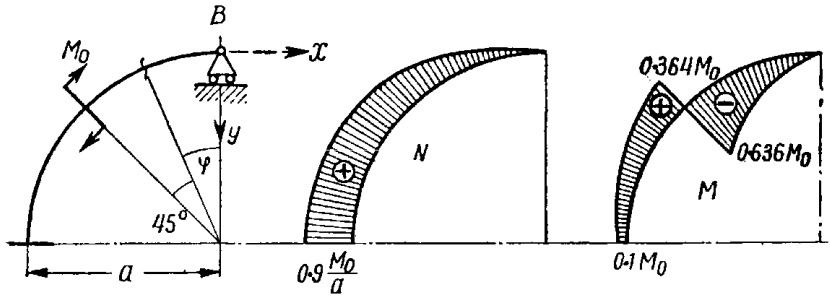
6.108. Solve the preceding problem, with the condition that the force is applied at the same point, but its direction is horizontal (from left to right).

Answer: $V_A = -V_B = H_A = H_B = \frac{P}{2}$;

$$\max M = 0{\cdot}207Pa;$$

$$\max N = 0{\cdot}707P.$$

6.109. Determine the reaction at support B, construct the M and N diagrams, and find the horizontal displacement of point B of the curved bar shown in the figure. (In determining the defor-



PROBLEM 6.109

mations and the reactions at the supports the normal force should not be taken into account.)

Answer: $B = 0{\cdot}9\,\dfrac{M_0}{a}$; $\quad x_B = 0{\cdot}042\,\dfrac{M_0 a^2}{EJ}$.

6.110. Determine the magnitude of the vertical and horizontal components of reaction B (see the preceding problem), with the assumption that support B is fixed.

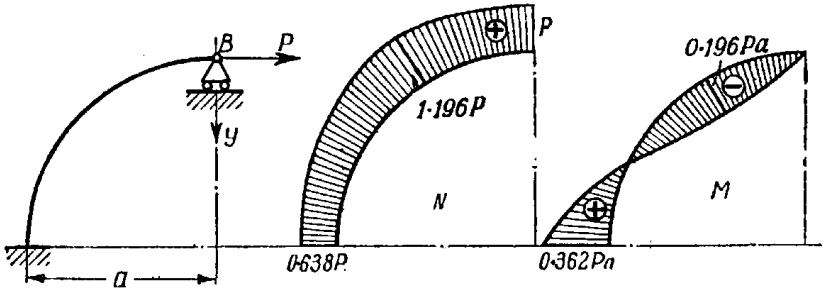
Answer: $B_y = 0{\cdot}19\,\dfrac{M_0}{a}$; $\quad B_x = 1{\cdot}12\,\dfrac{M_0}{a}$.

6.111. Find the reaction of support B, the maximum bending moment, and the maximum normal force (plot diagrams) for the curved bar represented in the figure. Determine the horizontal displacement of point B.

Answer: $B_y = \dfrac{2}{\pi}P$; $\quad \min M = 0{\cdot}196Pa$;

$\max M = 0{\cdot}362Pa$;

$\max N = 1{\cdot}196P$; $\quad x_B = 0{\cdot}038\,\dfrac{Pa^3}{EJ}$.

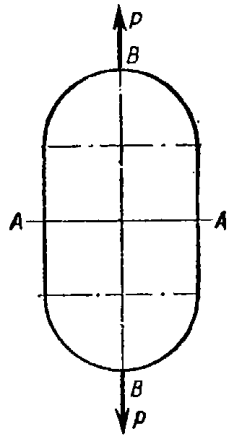


PROBLEM 6.111

6.112. A link made up of two circular half-rings and two straight portions is pulled by two forces P as shown in the figure. Find the bending moments in the sections coinciding with the axes of symmetry.

Answer: $$M_A = -\frac{\pi - 2}{2\dfrac{l}{a} + \pi}\,\frac{Pa}{2};$$

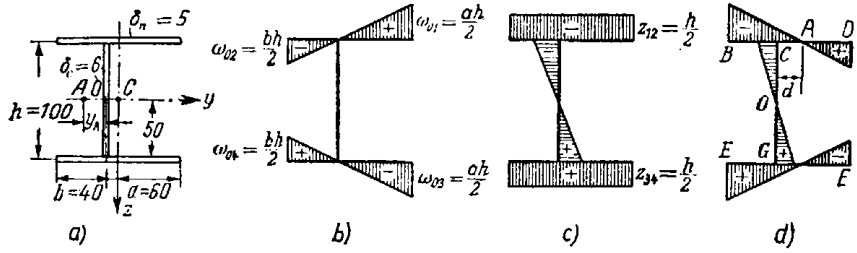
$$M_B = \left[1 - \frac{\pi - 2}{2\dfrac{l}{a} + \pi}\right]\frac{Pa}{2}.$$



PROBLEM 6.112

30. THIN-WALLED BARS *

6.113. Determine the position of the shear centre, draw a diagram of the principal sectorial areas, and calculate the value of the sectorial second moment of area of the profile shown in figure (a).



PROBLEM 6.113

* See also Appendix B, Section 4.

Solution. Since the cross-section is symmetrical with respect to the y axis the shear centre A will lie on this axis, i.e. $z_A = 0$. To find the distance y_A from the centre line of the vertical web of the profile to the shear centre we shall use the formula

$$y_A = \frac{S_{\omega_0 z}}{J_y},$$

where $S_{\omega_0 z}$ is the sectorial-linear first moment of area of the cross-section with respect to an arbitrarily chosen auxiliary pole and the principal centroidal z axis, and J_y is the second moment of area of the cross-section about the principal centroidal y axis.

We shall choose the auxiliary pole at point 0 (the intersection of the y axis with centre line of the web of the profile). Taking the same point as the origin we shall construct the diagram of the sectorial areas ω_0 (see figure (b)).

The integral $S_{\omega_0 z} = \int_F \omega_0 z \, dF$ can be calculated by Veresh-chagin's method, multiplying the diagram areas by the ordinates of the z diagram, the lines of which pass through the centroids of the areas. The distance z to the centre line of the flanges is constant, namely $z_{12} = -(h/2)$ for the top flange, and $z_{24} = +(h/2)$ for the bottom flange. The z diagram is represented in figure (c). Thus, for the part of the top flange located on the right from the web we have

$$\delta_{fl} \times \frac{1}{2} \omega_{01} a z_{12} = \delta_{f1} \times \frac{1}{2} \frac{ah}{2} a \left(-\frac{h}{2} \right) = -\frac{a^2 h^2}{8} \delta_{fl}.$$

The sectorial-linear first moments of area of the remaining parts of the cross-section can be calculated in a similar way; therefore for the whole cross-section we have

$$S_{\omega 0z} = \frac{1}{2} \delta_{fl} [\omega_{01} a z_{12} + \omega_{02} b z_{12} + \omega_{03} a z_{34} + \omega_{04} b z_{34}]$$

$$= \frac{1}{2} \delta_{fl} \left[-\frac{a^2 h^2}{4} + \frac{b^2 h^2}{4} - \frac{a^2 h^2}{4} + \frac{b^2 h^2}{4} \right] = \frac{h^2}{4} \delta_{f1} (a^2 - b^2)$$

$$= -\frac{10^2}{4} \times 0{\cdot}5 (6^2 - 4^2) = -250 \text{ cm}^5.$$

The second moment of area of the cross-section about the y axis is

$$J_y = 2(a+b)\,\delta_{fl}\left(\frac{h}{2}\right)^2 + (\delta_{web} \times \frac{h^3}{12}) = 2(6+4) \times 0{\cdot}5\left(\frac{10}{2}\right)^2$$

$$+ \left(0{\cdot}6 \times \frac{10^3}{12}\right) = 300 \text{ cm}^4.$$

Thus the co-ordinate y_A of point A (see figure (a)) is

$$y_A = \frac{S_{\omega_0 z}}{J_y} = -\frac{250}{300} = -0{\cdot}833 \text{ cm}.$$

Taking now point A as the principal sectorial pole, and point O as the principal sectorial zero point, we construct the diagram of the principal sectorial co-ordinates ω (see figure (d)). Thus

$$\omega_C = -y_A \times \frac{h}{2} = -4{\cdot}17 \text{ cm}^2; \quad \omega_B = \omega_C - \frac{bh}{2} = -24{\cdot}17 \text{cm}^2;$$

$$\omega_D = \omega_C + \frac{ah}{2} = +25{\cdot}83 \text{ cm}^2; \quad \omega_G = +4{\cdot}17 \text{ cm}^2;$$

$$\omega_E = +24{\cdot}17 \text{ cm}^2, \quad \text{and} \quad \omega_F = -25{\cdot}83 \text{ cm}^2.$$

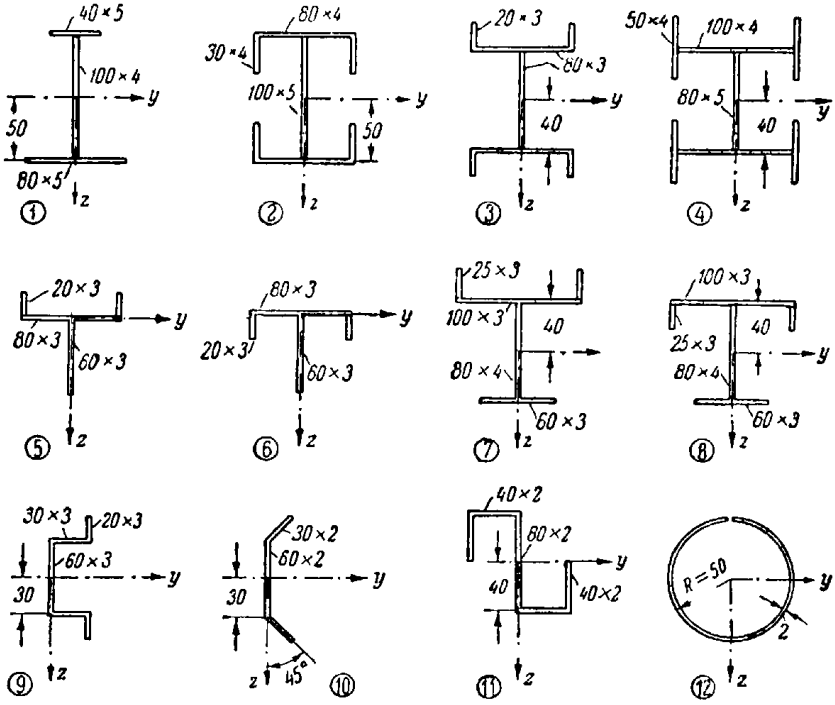
To calculate the sectorial second moment of area $J_\omega = \int_F \omega^2 dF$ we apply again Vereshchagin's method. For example, for portion HD of the top flange

$$J_\omega(\text{HD}) = \frac{1}{2}\,\omega_D(a-d) \times \frac{2}{3}\,\omega_D\,\delta_{fl} = \frac{1}{3}\,\omega_D^2(a-d)\delta_{fl}.$$

The calculation of J_ω is now carried out in a similar way for the rest of the parts of the cross-section. Thus

$$J_\omega = \frac{1}{3}\omega_D^2(a-d)\delta_{fl} + \frac{1}{3}\omega_D^2(b+d)\delta_{fl} + \frac{1}{6}\omega_C^2 h\delta_{web}$$
$$+ \frac{1}{6}\omega_G^2 h\delta_{web} + \frac{1}{3}\omega_E^2(b+d)\delta_{fl} + \frac{1}{3}\omega_F^2(a-d)\delta_{fl}$$
$$= \frac{2}{3}\left\{\delta_{fl}[\omega_D^2(a-d) + \omega_B^2(b+d)] + \omega_C^2\frac{h}{2}\delta_{web}\right\}$$
$$= \frac{2}{3}\left\{0{\cdot}5\ [(25{\cdot}83^2 \times 5{\cdot}17) + (24{\cdot}17^2 \times 4{\cdot}83)]\right.$$
$$\left. + \left(4{\cdot}17^2 \times \frac{10}{2} \times 0{\cdot}6\right)\right\} = 2125\ \text{cm}^6.$$

6.114. Determine the position of the shear centre (point A), construct a diagram of the principal sectorial areas, determine the maximum ordinate (ω_{max}) of this diagram, and calculate the sectorial second moment of area of the sections represented in the figure.



PROBLEM 6.114

Answer: (1) $y_A = 0$; $z_A = +3{\cdot}88$ cm; $|\omega_{max}| = 17{\cdot}76$ cm^2; $J_\omega = 237$ cm^6;

(2) $y_A = z_A = 0$; $|\omega_{max}| = 32$ cm^2; $J_\omega = 4156$ cm^6;

(3) $y_A = z_A = 0$; $|\omega_{max}| = 16$ cm^2; $J_\omega = 768$ cm^6;

(4) $y_A = z_A = 0$; $|\omega_{max}| = 32{\cdot}5$ cm^2; $J_\omega = 4683$ cm^6;

(5) $y_A = 0$; $z_A = +0{\cdot}6$ cm; $|\omega_{max}| = 5{\cdot}6$ cm^2; $J_\omega = 14{\cdot}1$ cm^6;

(6) $y_A = 0$; $z_A = -0{\cdot}6$ cm; $|\omega_{max}| = 5{\cdot}6$ cm^2; $J_\omega = 14{\cdot}1$ cm^6;

(7) $y_A = 0$; $z_A = -2{\cdot}67$ cm; $|\omega_{max}| = 20$ cm^2; $J_\omega = 304$ cm^6;

(8) $y_A = 0$; $z_A = -4{\cdot}051$ cm; $|\omega_{max}| = 24{\cdot}15$ cm^2; $J_\omega = 423{\cdot}5$ cm^6;

(9) $y_A = 1{\cdot}259$ cm; $z_A = 0$; $|\omega_{max}| = 5{\cdot}22$ cm^2; $J_\omega = 30{\cdot}0$ cm^6;

(10) $y_A = -0{\cdot}707$ cm; $z_A = 0$; $|\omega_{max}| = 2{\cdot}74$ cm^2; $J_\omega = 4{\cdot}28$ cm^6;

(11) $y_A = z_A = 0$; $|\omega_{max}| = 21{\cdot}3$ cm^2; $J_\omega = 547$ cm^6;

(12) $y_A = 0$; $z_A = +10$ cm; $|\omega_{max}| = 78{\cdot}5$ cm^2; $J_\omega = 5065$ cm^6.

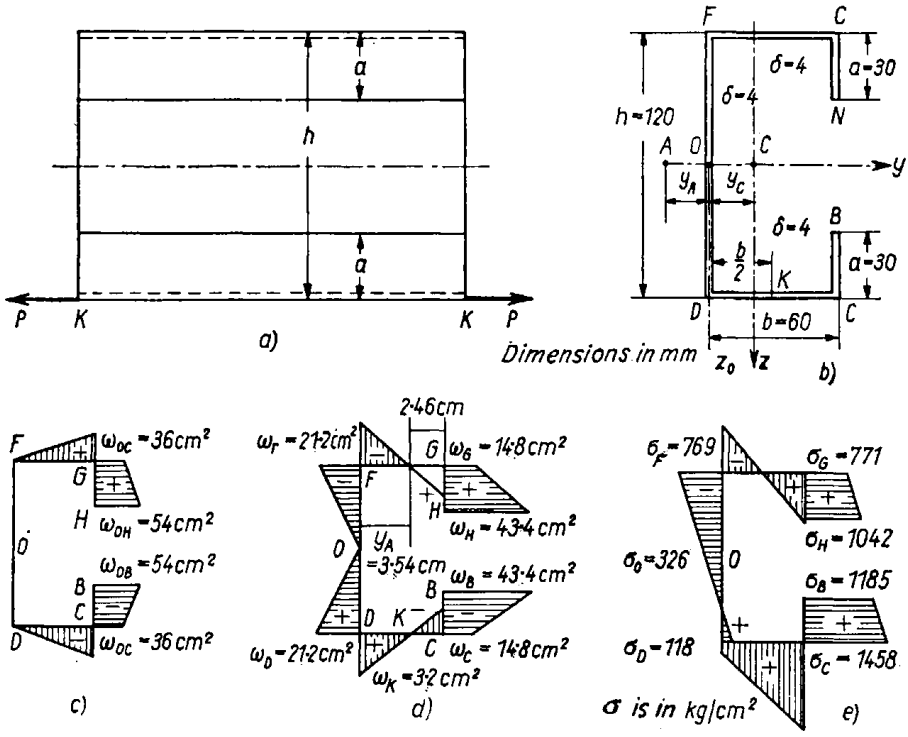
6.115. A bar with the cross-section shown in figure (b) is pulled by two forces $P = 3$ t applied to points K.

Determine the value of the normal stress at points B, C, D, O, F, G, and H, and construct diagrams showing the variation of this stress over the cross-section of the bar.

Solution. The geometrical characteristics of the cross-section. The area F is 12 cm^2. The co-ordinate of the centroid C of the cross-section from the z_0 axis is

$$y_C = \frac{S_{z_0}}{F} = \frac{b\,\delta(b + 2a)}{F} = 2{\cdot}4 \text{ cm}.$$

The second moment of area J_y of the cross-section is 280·8 cm^4. The second moment of area J_z of the cross-section about the z axis is 74·9 cm^4.



PROBLEM 6.115

To find the position of the shear centre an auxiliary diagram of sectorial areas is constructed, with the pole and the origin at point O (figure (c)). The distance y_A of the shear centre from the centre line of the web of the profile (figure (b)) is calculated from the formula

$$y_A = \frac{S_{\omega_o z}}{J_y} = \frac{994}{280{\cdot}8} = -3{\cdot}54 \text{ cm}.$$

The diagram (figure (d)) of the principal sectorial co-ordinates ω is constructed, taking point A as the principal sectorial pole, and point O as the principal sectorial zero point.

The sectorial second moment of area J_ω, calculated by Vereshchagin's method, is 3480 cm^6.

The calculation of the normal stress is carried out by the formula

$$\sigma = \frac{P}{F} + \frac{P y_p y}{J_z} + \frac{P z_p z}{J_y} + \frac{B\omega}{J_\omega}.$$

The magnitude of the "bi-moment" B at the sections where the forces P are applied is

$$B = B_{\max} = P\omega_K = P\left(\omega_D - \frac{h}{2} \times \frac{b}{2}\right) = 3000\left(21{\cdot}2 - \frac{12}{2} \times \frac{6}{2}\right)$$
$$= 9600 \text{ kg cm}^2.$$

Since

$$y_p = \frac{b}{2} - y_C = \frac{6}{2} - 2{\cdot}4 = 0{\cdot}6 \text{ cm} \quad \text{and} \quad z_p = \frac{h}{2} = \frac{12}{2} = 6 \text{ cm},$$

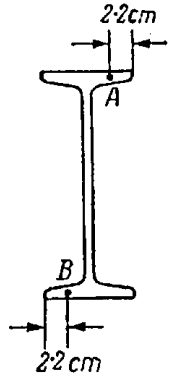
then

$$\sigma = \frac{3000}{12} + \frac{3000 \times 0{\cdot}6 \times y}{74{\cdot}9} + \frac{3000 \times 6 \times z}{280{\cdot}8} + \frac{9600 \times \omega}{3480}$$
$$= 250 + 24\, y + 64{\cdot}1\, z + 2{\cdot}76\omega.$$

The values of the normal stress, calculated by the above formula for points B, C, D, O, F, G and H, are given in a diagram showing the distribution of normal stress over the cross-section of the bar (see figure (e)).

The maximum normal stress occurs at point C: $\sigma_C = 1458 \text{kg/cm}^2$.

6.116. A No. 16 I-profile is being pulled by two forces $P = 9$ t applied to its end sections. Determine the magnitude of the maximum tensile stress, and that of the maximum compressive stress in the cross-sections of the I-profile, assuming that (a) forces P are applied to the centroids of the end sections, and (b) forces $P/2$ are applied to the end sections at points A and B, the mid-points



PROBLEM 6.116

of the flange thickness (see figure). In solving the problem make use of the data on the standard rolled sections in Appendix A.

Answer: (a) $\max \sigma_{tens} = 345$ kg/cm^2 = const;

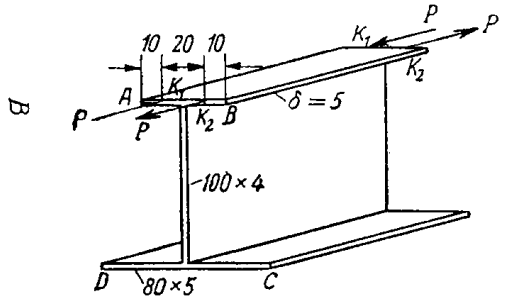
(b) $\max \sigma_{tens} = 1304$ kg/cm^2; $\max \sigma_{comp} = -614$ kg/cm^2.

6.117. A bar with the cross-section represented in the figure is loaded by forces $P = 1$ t applied to points K_1 and K_2.

Determine the value of the normal stress at points A, B, C, and D, and draw diagrams showing the distribution of this stress over the cross-section of the bar.

Answer: $\sigma_A = -1497$ kg/cm^2; $\sigma_B = +1497$ kg/cm^2;

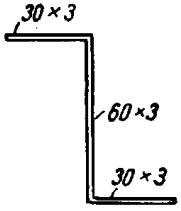
$\sigma_C = +3$ kg/cm^2; $\sigma_D = -3$ kg/cm^2.



PROBLEM 6.117

6.118. A bar of Z-section (see figure) is being pulled by centrally applied forces $P = 3$ t. Determine the maximum tensile and compressive stresses in the cross-section of the bar.

Answer: $+1333$ kg/cm^2; -667 kg/cm^2.



PROBLEM 6.118

6.119. Determine the value of the maximum shear stress, and the total angle of twist of the steel bar considered in Problem 6.115. One end of the bar is to be assumed clamped, and the other free of constraint. The length of the bar is 2 m. A tensile force $P = 3$ t

is applied to the free end of the bar at point K (see figure to Problem 6.115).

Answer: 50·6 kg/cm²; 0·0122 radians.

6.120. A No. 24 a channel 2·5 m long, with both ends clamped, is being twisted by a torque of 100 kgm applied to the middle of its length.

Determine the magnitude of the maximum sectorial normal stress, and the maximum shear stress of pure torsion. In solving the problem make use of the data on standard rolled sections given in Appendix A.

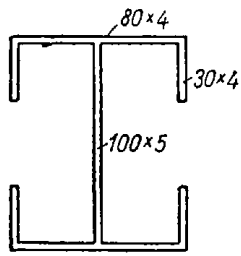
Answer: $\max \sigma_\omega = 807$ kg/cm²; $\max \tau_t = 189$ kg/cm².

6.121. A steel bar of the cross-section represented in the figure, clamped at one end, is being twisted by a torque of 20 kgm applied to the free end. The length of the bar is 1·6 m.

Determine the value of the maximum sectorial normal stress, the maximum sectorial shear stress, and the maximum shear stress of pure torsion, and find the angle of twist of the free end of the bar.

Answer:

$$\max \sigma_\omega = 1432 \text{ kg/cm}^2;$$
$$\max \tau_\omega = 56{\cdot}8 \text{ kg/cm}^2;$$
$$\max \tau_t = 597 \text{ kg/cm}^2;$$
$$\theta = 0{\cdot}165 \text{ radians}.$$



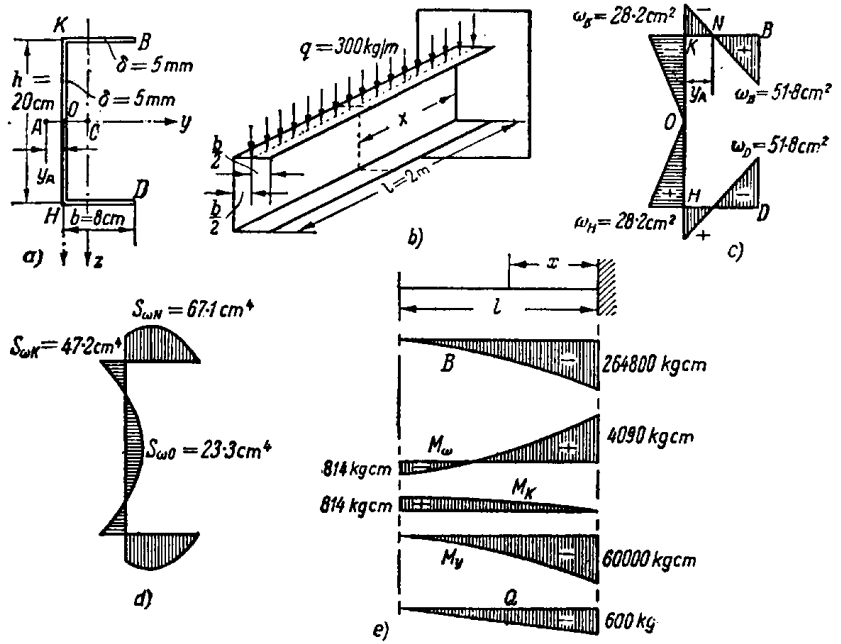
PROBLEM 6.121

6.122. A No. 20 a I-beam 1·8 m long, clamped at one end, is being bent in the plane of the web by a concentrated force of 500 kg applied to the free end, and twisted by a torque of 60 kgm applied to the middle of its length. Determine the value of the maximum normal stress and the maximum shear stress at the constraint and

at the mid-point of the length of the beam. Determine also the value of the maximum sectorial shear stress at the free end of the beam. In solving the problem make use of the data on standard rolled sections given in Appendix A.

Answer: At the free end of the beam: $\max \tau_\omega = 5{\cdot}5\ \text{kg/cm}^2$; at the middle section: $\tau_{\max} = 342\ \text{kg/cm}^2$ and $\sigma_{\max} = 190\ \text{kg/cm}^2$; at the constraint: $\tau_{\max} = 62{\cdot}8\ \text{kg/cm}^2$ and $\sigma_{\max} = 1350\ \text{kg/cm}^2$.

6.123. A bar of channel section (see figure (a)), with a length l ($= 2$ m), is clamped at one end and loaded by a uniformly distributed load of intensity q ($= 300$ kg/m), applied to the middle of the flange width (see figure (b)). Determine the value of the maximum normal stress, and that of the maximum shear stress.



PROBLEM 6.123

Solution. The geometrical characteristics of the cross-section of the bar are: the cross-sectional area $F = 18\ \text{cm}^2$; the second moment of area $J_y = 1133\ \text{cm}^4$; the second moment of area for pure torsion $J_t = 1{\cdot}5\ \text{cm}^4$. The co-ordinate of the shear centre $y_A = -2{\cdot}82$ cm. The sectorial second moment of area of the cross-section $J_\omega = 8030\ \text{cm}^6$.

The diagram of the sectorial first moment of area, calculated as the sum of the areas of the ω diagram multiplied by the respective flange and web thickness, is represented in figure (d). The maximum sectorial first moment of area is

$$\max S_\omega = S_{\omega N} = \frac{\delta \omega_B (b - y_A)}{2} = 67{\cdot}1 \text{ cm}^4.$$

The ordinary first moment of area about the y axis of the part of the flange, on the right from point N, is $S_{yN} = 25{\cdot}9 \text{ cm}^3$, and the first moment of area of one half of the cross-section about the same axis is $S_{y_0} = 65 \text{ cm}^3$.

The elastic flexuro-torsional characteristic of the bar is

$$\alpha = \sqrt{\frac{GJ_t}{EJ_\omega}} = \sqrt{\frac{8 \times 10^5 \times 1{\cdot}5}{2 \times 10^6 \times 8030}} = \sqrt{0{\cdot}00007472}$$

$$= 0{\cdot}008644 \text{ cm}^{-1};$$

$\alpha l = 0{\cdot}008644 \times 200 = 1{\cdot}729$; $\sinh \alpha l = 2{\cdot}729$; $\cosh \alpha l = 2{\cdot}907$.

We shall now determine the force factors. The solution of the differential equation has the form

$$B = \frac{qe}{\alpha^2}\left[\alpha l \sinh \alpha x - \frac{1 + \alpha l \sinh \alpha l}{\cosh \alpha l} \cosh \alpha x + 1\right],$$

$$M_\omega = \frac{dB}{dx} = \frac{qe}{\alpha}\left[\alpha l \cosh \alpha x - \frac{1 + \alpha l \sinh \alpha l}{\cosh \alpha l} \sinh \alpha x\right],$$

and

$$M_t = M_0 - M_\omega = qe(l - x) - M_\omega$$

$$= \frac{qe}{\alpha}\left[-\alpha l \cosh \alpha x + \frac{1 + \alpha l \sinh \alpha l}{\cosh \alpha l} \sinh \alpha x + \alpha(l - x)\right].$$

The graphs of B, M_ω and M_t are represented in figure (e).

B and M assume their maximum values at the constraint, where at $x = 0$,

$$B_{\max} = \frac{qe}{\alpha^2}\left(1 - \frac{1 + \alpha l \sinh \alpha l}{\cosh \alpha l}\right)$$

$$= \frac{3 \times 6{\cdot}82}{0{\cdot}00007472}\left[1 - \frac{1 + (1{\cdot}729 \times 2{\cdot}729)}{2{\cdot}907}\right]$$

$$= -264{,}800 \text{ kg cm}^2$$

and

$$M_{\omega \max} = qel = 3 \times 6{\cdot}82 \times 200 = 4090 \text{ kg cm}.$$

The torque causing pure shear reaches the maximum value near the middle of the length; the torque at the free end differs very little from this, i.e. at $x = l$,

$$M_t = \frac{qe}{\alpha} \times \frac{\sinh \alpha l - \alpha l}{\cosh \alpha l} = \frac{3 \times 6{\cdot}82}{0{\cdot}008644} \times \frac{2{\cdot}729 - 1{\cdot}729}{2{\cdot}907}$$

$$= 814 \text{ kg cm}.$$

At the same cross-section

$$M_\omega = \frac{qe}{\alpha} \times \frac{\sinh \alpha l - \alpha l}{\cosh \alpha l} = -M_{t\max} = -814 \text{ kg cm}.$$

Besides the action of the force factors enumerated, the bar is subjected to plane flexure in the vertical plane. The bending moment $M_y = q(l - x)^2/2$; the shear force $Q = -q(l - x)$. The maximum values are reached by M_y and Q at the constraint of the bar, where

$$\max M_y = -\frac{ql^2}{2} = -\frac{3 \times 200^2}{2} = -60{,}000 \text{ kg cm}$$

and

$$\max Q = -ql = -(3 \times 200) = -600 \text{ kg}.$$

The normal stress at the constraint is calculated by the formula

$$\sigma = \frac{M_y z}{J_y} + \frac{B_\omega}{J_\omega} = -\frac{60{,}000z}{1133} - \frac{264{,}800\omega}{8030} = 53z - 33\omega.$$

The values of the co-ordinates z and ω, and the normal stress at points B, D, H, and K of the cross-section are given in the table

Point	Z (cm)	ω (cm^2)	σ (kg/cm^2)
B	-10	$+51{\cdot}8$	-1180
D	$+10$	$-51{\cdot}8$	$+1180$
H	$+10$	$+28{\cdot}2$	-1460
K	-10	$-28{\cdot}2$	$+1460$

The shear stress at the constraint consists of components due to the shear force and the flexural torque M_ω:

$$\tau = \frac{QS_y}{\delta J_y} + \frac{M_\omega S_\omega}{\delta J_\omega}.$$

The maximum value is reached by the shear stress at point N, where the directions of all shear stresses coincide:

$$\tau_{\max} = \tau_N = \frac{600 \times 25{\cdot}9}{0{\cdot}5 \times 1133} + \frac{4090 \times 67{\cdot}1}{0{\cdot}5 \times 8030}$$
$$= 27{\cdot}4 + 68{\cdot}4 = 96 \text{ kg/cm}^2.$$

while at point O

$$\tau_0 = \frac{600 \times 65}{0{\cdot}5 \times 1133} - \frac{4090 \times 23{\cdot}3}{0{\cdot}5 \times 8030} = 68{\cdot}7 - 23{\cdot}7 = 45 \text{ kg/cm}^2.$$

At the free end of the bar the shear stress consists of components due to the pure torsion and the flexural torque:

$$\tau = \frac{M_t \delta}{J_t} + \frac{M_\omega S_\omega}{\delta J_\omega}.$$

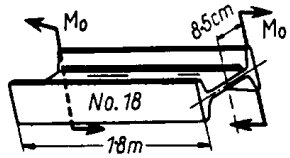
Here the shear stress assumes its maximum value at point N, where

$$\tau_{\max} = \tau_N = \frac{814 \times 0{\cdot}5}{1{\cdot}5} + \frac{814 \times 67{\cdot}1}{0{\cdot}5 \times 8030} = 271 + 14 = 285 \text{ kg/cm}^2.$$

6.124. A No. 18 I-beam, the ends of which are restrained from rotation about the longitudinal axis, but which can warp freely, is loaded by a torque $M_0 = 200$ kg m acting in the plane of one of the flanges (see figure). The length of the I-beam is 1·8 m.

Determine the value of the maximum normal stress and the maximum shear stress in the cross-section of the I-profile. In solving the problem make use of the data on standard rolled sections given in Appendix A. $E = 2{\cdot}1 \times 10^6$ kg/cm^2; $G = 8 \times 10^5$ kg/cm^2.

Answer: $\sigma_{max} = 1574$ kg/cm^2; $\tau_{max} = 319$ kg/cm^2.



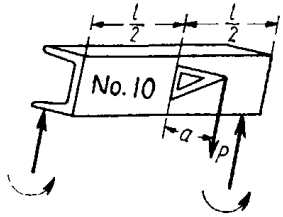
PROBLEM 6.124

6.125. A No. 10 channel, the ends of which are restrained from rotation about the longitudinal axis, but which may warp freely, is loaded at the middle of the span $l = 1{\cdot}5$ m by a concentrated force $P = 200$ kg applied as shown in the figure. The distance a is 25 cm.

Determine the value of the maximum normal stress and the maximum shear stress in the cross-section of the channel. In solving the problem make use of the data on standard rolled sections given in Appendix A.

$$E = 2{\cdot}1 \times 10^6 \text{ kg/cm}^2; \quad G = 8 \times 10^5 \text{ kg/cm}^2.$$

Answer: $\sigma_{max} = 319$ kg/cm^2; $\tau_{max} = 766$ kg/cm^2.



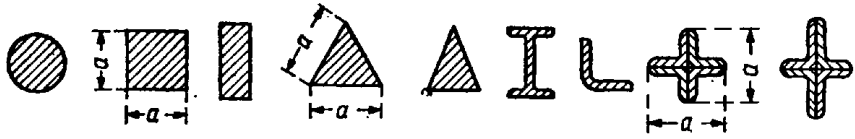
PROBLEM 6.125

CHAPTER 7

STABILITY OF STRUCTURAL ELEMENTS

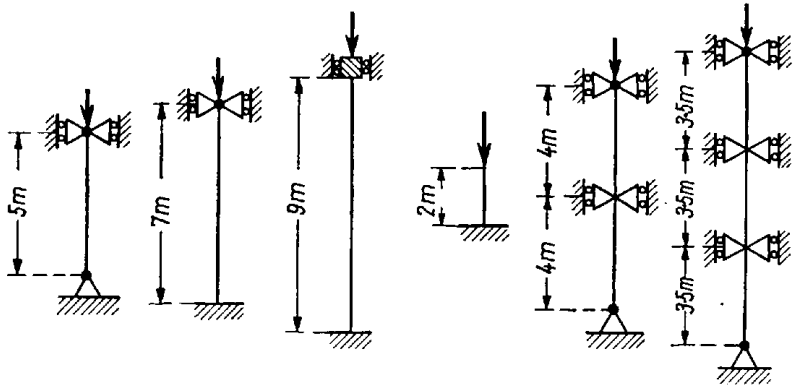
31. STABILITY OF RODS IN COMPRESSION

7.1. The cross-sections of several rods are shown in the figure. In which direction will they buckle?



PROBLEM 7.1

7.2. Which rod (see figure) will withstand the greatest, and which the smallest longitudinal compressive load, if the cross-section of the rods is the same?



PROBLEM 7.2

7.3. Determine the minimum slenderness ratio at which Euler's formula is still applicable for the calculation of the critical force, if the rod is made of:

(a) steel, with a limit of proportionality $\sigma_p = 2200$ kg/cm^2, and a modulus of elasticity $E = 1{\cdot}9 \times 10^6$ kg/cm^2;

(b) steel, with $\sigma_p = 820$ kg/cm^2, and $E = 1{\cdot}85 \times 10^6$ kg/cm^2;

(c) 3·5% nickel steel, with $\sigma_p = 4900$ kg/cm^2, and $E = 2{\cdot}15 \times 10^6$ kg/cm^2;

(d) duralumin, with $\sigma_p = 1770$ kg/cm^2, and $E = 0{\cdot}7 \times 10^6$ kg/cm^2;

(e) pinewood, with $\sigma_p = 200$ kg/cm^2, and $E = 1{\cdot}1 \times 10^5$ kg/cm^2.

Answer: (a) 92·5; (b) 149; (c) 65·8; (d) 62·5; (e) 73·7.

7.4. Using Euler's formula, determine the value of the critical load, and the critical stress for a column in compression having the cross-section of a No. 22a I-profile. The column is pinned at both ends (spherical joints). The length of the column is 5 m. The material is steel having a limit of proportionality $\sigma_p \approx 2000$ kg/cm^2.

Solution. A No. 22a I-section has $F = 42$ cm^2, $J_{\min} = J_y = 225$ cm^4 and $i_{\min} \cdot = i_y = 2{\cdot}31$ cm. The slenderness ratio of the column is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{1 \times 500}{2{\cdot}31} = 216 > \lambda_{\lim} \approx 100.$$

Thus, Euler's formula is applicable. The critical force is

$$P_{\text{crit}} = \frac{E J_{\min} \pi^2}{(\mu l)^2} = \frac{2 \times 10^6 \times 225 \times 9{\cdot}87}{(1 \times 500)^2} = 17{,}770 \text{ kg}.$$

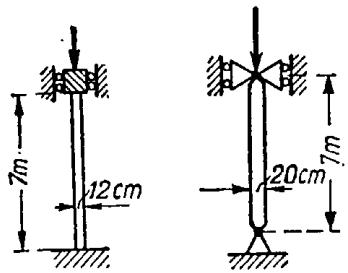
The critical stress is

$$\sigma_{\text{crit}} = \frac{P_{\text{crit}}}{F} = \frac{17{,}770}{42} = 423 \text{ kg/cm}^2.$$

7.5. Determine the magnitude of the critical force and the critical stress for a column 7 m long, having a rectangular cross-section

of 12 × 20 cm². It is made of wood having a modulus of elasticity $E = 0{\cdot}9 \times 10^5$ kg/cm². In the plane of the minimum stiffness both ends of the column are clamped, but in the plane perpendicular to it both ends of the column are hinged (see figure).

Answer: 14,500 kg; 60·4 kg/cm².



PROBLEM 7.5

7.6. Using Euler's formula choose a standard I-section for a column 2·2 m long subjected to a longitudinal compressive load $P = 11$ t. One end of the column is clamped; the other end is pinned. The material is Steel 3. Factor of safety in buckling $k_s = 2{\cdot}5$.

Solution. Since

$$P \leq \frac{P_{crit}}{k_s} = \frac{EJ_{min}\pi^2}{k_s(\mu l)^2},$$

then

$$J_{min} \geq \frac{k_s P(\mu l)^2}{E\pi^2} = \frac{2{\cdot}5 \times 11{,}000 \times (0{\cdot}7 \times 220)^2}{2 \times 10^6 \times 9{\cdot}87} = 34 \text{ cm}^4.$$

According to the tables, a No. 10 I-section has the required second moment of area $J_{min} = J_y$. The radius of gyration of the cross-section of this I-profile $i_{min} = i_y = 1{\cdot}52$ cm. The effective slenderness ratio of the column is

$$\lambda = \frac{\mu l}{i_{min}} = \frac{0{\cdot}7 \times 220}{1{\cdot}52} = 101{\cdot}3.$$

With this slenderness ratio, it is permissible to use Euler's formula for the calculation.

7.7. Using Euler's formula find the ratio of the critical load on a column of circular cross-section to that on a column of annular cross-section of ratio $d_{ext}: d_{int} = 1{\cdot}25$, if they work in identical conditions and have equal cross-sectional areas.

Answer: $P_{circ}: P_{ann} = 0{\cdot}22$.

7.8. Which of the two rods, with identical constraints and loading, is more flexible, a rod of square cross-section, or a rod of circular cross-section having the same cross-sectional area?

Answer: The slenderness ratio of the second rod is 2·3% greater.

7.9. A straight bar 1 m long, 2·5 cm wide and 2·5 mm thick is bent in the shape of a bow, the elastic deflection at the middle being 5 cm. Its ends are joined by a bow-string. Assuming the modulus of elasticity, E, of the material of the bar to be 2×10^6 kg/cm², determine the force in the bow-string and the maximum stress in the bar.

Answer: $P \approx P_{crit} = 6{\cdot}45$ kg; $\sigma_{max} = 1250$ kg/cm² (for a deflection of 5 cm).

7.10. A No. 24a I-section, 5 m long, is built into a wall at both ends. At what temperature will the I-beam buckle, if the walls are assumed to be (a) absolutely rigid, (b) flexible, each having a compliance of 2×10^{-6} cm/kg $(= k)$?

Answer: (a) 89°C; (b) 145°C.

7.11. Determine the dimensions of the cross-section of an annular column, made of duralumin, with a ratio of the internal diameter to the external diameter equal to 0·8. For the given grade of duralumin, according to experimental data, $\sigma_{crit} = 3360 - 28\lambda$ (kg/cm²) for $0 \leq \lambda \leq 80$. The length of the column is 1·8 m; both of its ends are clamped. The load is 7 t; $k_s = 3$.

Answer: $d_0 = 60$ mm; $d_1 = 48$ mm.

7.12. Making use of the table of allowable stress factors (see Appendix A), determine the maximum allowable value of a compressive load on an I-section column (I No. 24a), made of Steel 3. The basic allowable stress $[\sigma]$ is 1600 kg/cm². The length of the column is 2·5 m; both its ends are pinned.

Solution. A No. 24a I-section, according to the tables of standard sections, has $F = 47{\cdot}7\ \text{cm}^2$ and $i_{\min} = 2{\cdot}42\ \text{cm}$. The effective slenderness ratio of the column is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{1 \times 250}{2{\cdot}42} = 103{\cdot}3.$$

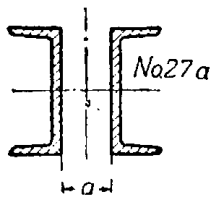
From the table for Φ, the allowable stress factor in longitudinal flexure, we find that $\Phi = 0{\cdot}574$ (after a linear interpolation).

The allowable force is

$$[P] = \Phi[\sigma]F = 0{\cdot}574 \times 1600 \times 47{\cdot}7 = 43{,}800\ \text{kg}.$$

7.13. Determine the maximum allowable value of a compressive load on a column consisting of two No. 27a channels set apart so that the centroidal second moments of area about both principal axes are equal (see figure). The material is Steel 3; the basic allowable stress $[\sigma]$ is 1600 kg/cm². The length of the column is 6 m; one of its ends is clamped, while the other is free of constraint. Determine the required minimum value of dimension a.

Answer: 62 t; 16·3 cm.



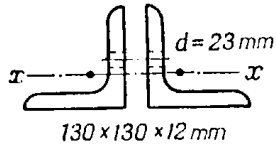
PROBLEM 7.13

7.14. Determine the maximum allowable value of a compressive load on a wooden post having a square cross-section of $15 \times 15\ \text{cm}^2$, if the basic allowable stress is 110 kg/cm². The length of the post is 3·5 m; both of its ends are pinned.

Answer: 11·7 t.

7.15. Check the stability and the strength of an element of an arched truss subjected to compression. It consists of two equal 130 mm × 130 mm × 12 mm angles (see figure) and has a length of 2·4 m. Both ends of the element are pinned; the load is 80 t.

The basic allowable stress $[\sigma]$ is 1600 kg/cm². The diameter of the rivet holes is 23 mm.



PROBLEM 7.15

Solution. Each of the angles of which the element of the truss is made up has $F = 30$ cm², $J = 477$ cm⁴ and $i_x = 3{\cdot}99$ cm. For the whole cross-section of the element $F_{\text{gross}} = 2 \times 30 = 60$ cm² and $i_{\min} = i_x = 3{\cdot}99$ cm. The effective slenderness ratio of the element is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{1 \times 240}{3{\cdot}99} = 60{\cdot}2.$$

The corresponding value of the factor reducing the allowable stress is found from the table: $\Phi = 0{\cdot}859$. The allowable load is

$$[P] = \Phi[\sigma]F_{\text{gross}} = 0{\cdot}859 \times 1600 \times 60 = 82{,}500\,\text{kg} > P = 80{,}000\,\text{kg}.$$

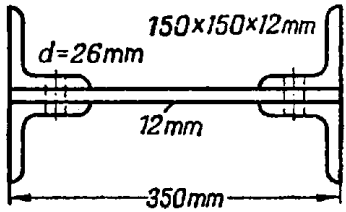
We shall check the strength of the element. The rivet holes reduce the strength of the section by $\Delta F = 2 \times 1{\cdot}2 \times 2{\cdot}3 = 5{\cdot}52$ cm². The compressive stress in the cross-section thus weakened is

$$\sigma = \frac{P}{F_{\text{net}}} = \frac{80{,}000}{60 - 5{\cdot}52} = 1468\,\text{kg/cm}^2 < [\sigma] = 1600\,\text{kg/cm}^2.$$

Thus, for a load of 80 t, the stability and the strength of the element are adequate.

7.16. Proceeding from the checks of stability and strength, determine the maximum load-carrying capacity of a column in compression, having the cross-section shown in the figure. The length of the column is 7 m; both of its ends are pinned. The diameter of the rivet holes is 26 mm. The material is Steel 3, and the basic allowable stress $[\sigma]$ is 1600 kg/cm².

Answer: 130 t.

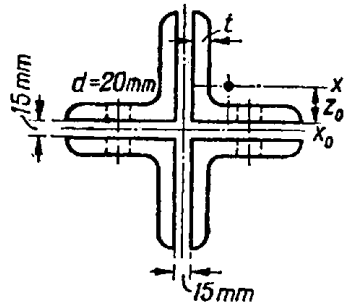


PROBLEM 7.16

7.17. Determine the cross-section of a column made up of equal angles (see figure). It is under a compressive load of 60 t. The length of the column is 3·5 m; its ends are pinned. The material is Steel 3; the basic allowable stress $[\sigma]$ is 1600 kg/cm².

Solution. We shall carry out the selection of the cross-section by means of successive approximations. In the first approximation we assume tentatively that $\Phi = 0{\cdot}5$. The required cross-sectional area of the column is

$$F_{\text{gross}} = \frac{P}{\Phi[\sigma]} = \frac{60{,}000}{1600 \times 0{\cdot}5} = 75 \text{ cm}^2.$$



PROBLEM 7.17

It is seen from the table of standard sections (see Appendix A) that this area may be obtained by taking four 100mm ×100mm ×10 mm angles, each with $F = 19{\cdot}20$ cm², $J_x = 179$ cm⁴ and $z_0 = 2{\cdot}83$ cm. Then the cross-sectional area of the column is

$$F_{\text{gross}} = 4 \times 19{\cdot}2 = 76{\cdot}8 \text{ cm}^2,$$

and

$$J_{x_0} = 4\left[179 + 19{\cdot}2\left(2{\cdot}83 + \frac{1{\cdot}5}{2}\right)^2\right] = 1700 \text{ cm}^4.$$

The radius of gyration of the cross-section is

$$i_{x_0} = \sqrt{\frac{J_{x_0}}{F}} = \sqrt{\frac{1700}{76{\cdot}8}} = 4{\cdot}71 \text{ cm}.$$

The effective slenderness ratio of the column is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{1 \times 350}{4{\cdot}71} = 74{\cdot}3.$$

From the table for Φ we find that $\Phi = 0{\cdot}784$. The allowable force $[P] = \Phi[\sigma]F_{\text{gross}} = 0{\cdot}784 \times 1600 \times 76{\cdot}8 = 96{,}300$ kg.

This force exceeds the actual load by more than one and a half times. The cross-section calculated will be grossly understressed. We assume now that $\Phi = 0{\cdot}7$. The required cross-sectional area $F_{\text{gross}} = 60{,}000/(0{\cdot}7 \times 1600) = 53{\cdot}5 \text{ cm}^2$.

We select a 75 mm × 75 mm × 10 mm angle having a cross-sectional area $F = 14{\cdot}1 \text{ cm}^2$, $J_x = 72{\cdot}2 \text{ cm}^4$ and $z_0 = 2{\cdot}22$ cm. Then for the whole cross-section of the column

$$F_{\text{gross}} = 4 \times 14{\cdot}1 = 56{\cdot}4 \text{ cm}^2$$

and

$$J_{x_0} = 4\left[72{\cdot}2 + 14{\cdot}1\left(2{\cdot}22 + \frac{1{\cdot}5}{2}\right)^2\right] = 786 \text{ cm}^4.$$

The radius of gyration of the cross-section is

$$i_{x_0} = \sqrt{\frac{786}{56{\cdot}4}} = 3{\cdot}73 \text{ cm}.$$

The effective slenderness ratio of the column is

$$\lambda = \frac{1 \times 350}{3{\cdot}73} = 93{\cdot}8.$$

From the table we find that $\Phi = 0{\cdot}656$. The allowable force is

$$[P] = \Phi[\sigma]F_{\text{gross}} = 0{\cdot}656 \times 1600 \times 56{\cdot}4 = 59{,}200 \text{ kg}.$$

We shall settle for the 75 mm × 75 mm × 10 mm angles, and check the strength of the column. The cross-section of the column, which is weakened by the rivet holes, has an area,

$$F_{\text{net}} = F_{\text{gross}} - 2dt = 56{\cdot}4 - (2 \times 2 \times 1) = 52{\cdot}4 \text{ cm}^2.$$

The compressive stress in the cross-section thus weakened is

$$\sigma = \frac{P}{F_{\text{net}}} = \frac{60{,}000}{52{\cdot}4} = 1145 \text{ kg/cm}^2 < [\sigma] = 1600 \text{ kg/cm}^2.$$

The strength of the proposed column is adequate.

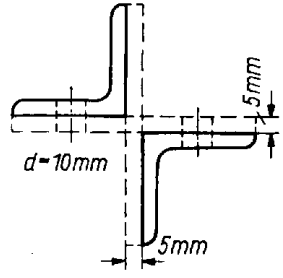
7.18. Select a standard I-section (see Appendix A) for a column 5·5 m long subjected to a compressive load of 23 t. Both ends of the column are clamped. The material is Steel 3. The basic allowable stress $[\sigma]$ is 1600 kg/cm².

Answer: I No. 20*a*.

7.19. Determine the cross-section of a diagonal member of a truss, made up of two equal angles (see figure), and subjected to a compressive load of 13·25 t. The length of the diagonal member is 2 m; its ends are pinned. The material is Steel 3; the basic allowable stress $[\sigma]$ is 1600 kg/cm².

Instruction. To calculate the second moment of area of the cross-section of the member, the angles should be assumed to be connected rigidly to each other. The increased flexibility, due to the bracing not being absolutely rigid, is not to be taken into consideration.

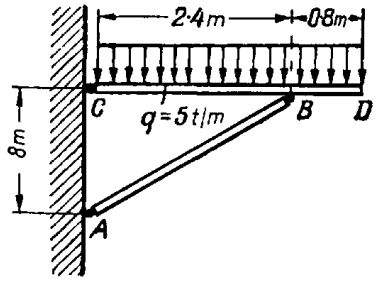
Answer: Two angles 60 mm × 60 mm × 5 mm.



PROBLEM 7.19

7.20. Determine the diameter of prop AB of a bracket (see figure), made of wood, if the basic allowable stress $[\sigma]$ is 110 kg/cm². Assume that both ends of the prop are pinned. The load q, uniformly distributed along the beam CD, is 5 t/m.

Answer: 18 cm.

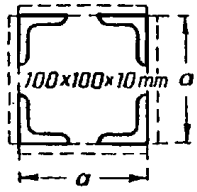


PROBLEM 7.20

7.21. A column consists of four angles 100mm × 100mm × 10mm connected by straps (see figure). Both ends of the column are pinned. Its length is 6 m, and it is compressed by a force of 100 t.

Determine the dimension, a, of the side of the cross-section of the column. The material is Steel 3; $[\sigma] = 1600$ kg/cm². When calculating the second moment of area of the cross-section of the column, assume that the angles are rigidly connected to each other. The increased flexibility, due to the bracing not being absolutely rigid, is not to be taken into consideration.

Answer: 21·9 cm.



PROBLEM 7.21

7.22. Determine the critical length of a column of circular cross-section 10 cm in diameter, clamped at one end and free of constraint at the other, for which it buckles under its own weight, if the column is made of (a) a steel having a specific weight $\gamma = 8$ g/cm³ and a modulus of elasticity $E = 2 \times 10^6$ kg/cm²; (b) duralumin,

with $\gamma = 2{\cdot}6$ g/cm^3 and $E = 0{\cdot}7 \times 10^6$ kg/cm^2; (c) timber, with $\gamma = 0{\cdot}6$ g/cm^3 and $E = 10^5$ kg/cm^2.

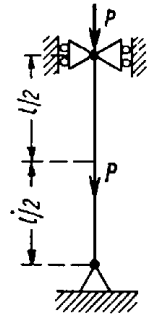
Use Yasinskii's formula: $(ql)_{\text{crit}} = (EJ_{\min}\pi^2)/(\mu l)^2$, where $\mu = 1{\cdot}12$.

Answer: (a) 23·1 cm; (b) 23·7 m; (c) 20·2 m.

7.23. A wooden column 24 cm in diameter and 6·5 m in length is loaded as indicated in the figure.

Determine the maximum allowable value of the load P. Apply Euler's formula. To calculate the length factor μ, depending on the loading conditions of the column (make use of the data of Table 30* in *Strength of Materials* by N. M. Belyayev, 1954 Edition). The factor of safety in buckling $k_s = 3$; $E = 10^5$ kg/cm^2.

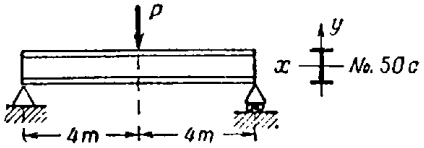
Answer: $P_{\max} = 8365$ kg.



PROBLEM 7.23

32. COMBINED STABILITY CALCULATIONS

7.24. Check the strength and the stability of the plane mode of flexure of a beam which is simply supported and loaded at the mid-span by a concentrated force $P = 10$ t. The cross-section of the beam is I No. 50c (see figure). The allowable stress $[\sigma]$ is 1600 kg/cm^2. The factor of safety in buckling k_s is 1·7.



PROBLEM 7.24

* See Appendix B, Section 6.

Solution. In checking the stability of the plane mode of flexure of the beam, the value of the allowable load is calculated by the formula

$$[P] = \frac{P_{\text{crit}}}{k_s} = \frac{\beta}{k_s l^2}\sqrt{C_1 C_2},$$

where C_1 and C_2 are respectively the flexural and torsional rigidities of the beam, and β is a coefficient depending on the type of loading, the end conditions of the beam, and the value of the ratio

$$\alpha = \frac{C_2}{C_1} \times \frac{l^2}{h^2},$$

where h is the depth of the cross-section of the beam.

In the given case

$$C_1 = EJ_{\min} = EJ_y = 2 \times 10^6 \times 1220 = 244 \times 10^7 \text{ kg cm}^2,$$

and

$$C_2 = GJ_t = G\frac{\eta}{3}[(h - 2t)d^3 + 2bt^3],$$

where b and t respectively are the length and the thickness of the flange of the I-section, d is the web thickness, and η is a factor dependent on the shape of the cross-section; for an I-section $\eta = 1{\cdot}2$. Thus

$$C_2 = 8 \times 10^5 \times \frac{1{\cdot}2}{3}\{[50 - (2 \times 2)] \times 1{\cdot}6^3 + (2 \times 16{\cdot}2 \times 2^3)\}$$
$$= 14{\cdot}32 \times 10^7 \text{ kg cm}^2.$$

The ratio

$$\alpha = \frac{14{\cdot}32 \times 10^7}{244 \times 10^7} \times \left(\frac{800}{50}\right)^2 = 15{\cdot}03.$$

By means of interpolation (from Table 33* in *Strength of Materials* by N. M. Belyayev, 1954 Edition) we find that for this value of α the coefficient $\beta = \beta_2 = 18{\cdot}39$. Then

$$[P] = \frac{18{\cdot}39}{1{\cdot}7 \times 800^2}\sqrt{244 \times 10^7 \times 14{\cdot}32 \times 10^7} = 9990 \text{ kg} \approx P =$$
$$= 10{,}000 \text{ kg}.$$

* See Appendix B, Section 6.

The check of the strength of the beam, with $W = 2080$ cm, gives

$$\sigma_{max} = \frac{M_{max}}{W_x} = \frac{Pl}{4W_x} = \frac{10{,}000 \times 800}{4 \times 2080} = 962 \text{ kg/cm}^2 < [\sigma]$$
$$= 1600 \text{ kg/cm}^2.$$

Thus in establishing the value of the allowable load one should proceed from the check of the stability of the beam.

7.25. Proceeding from the check of the strength and the stability of the plane mode of flexure of a steel cantilever, determine its maximum load-carrying capacity. The beam has a rectangular cross-section of 200×12 mm^2 (the depth 200 mm is parallel to the plane of action of the load) and carries a uniformly distributed load of intensity q over its full length. The length of the beam is 2 m; $[\sigma] = 1400$ kg/cm^2; $k_s = 1{\cdot}8$.

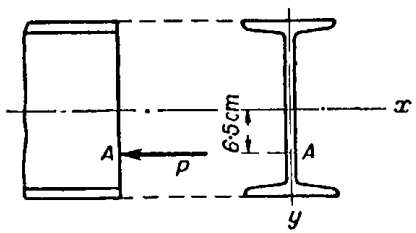
Answer: $q_{max} = 560$ kg/m.

7.26. A cantilever of welded construction, consisting of 150×20 mm^2 (the flanges) and 660×15 mm^2 (the web) steel plates, is loaded by a concentrated force P at its free end.

Determine the length of the beam and the magnitude of the force P from the condition that the strength and the stability of the plane mode of flexure of the beam are equal, if $[\sigma]$ is 1600 kg/cm^2 and k_s is 1·7.

Answer: 5·07 m; 9·5 t.

7.27. A simply supported No. 20b I-beam, 2·4 m long, is loaded by longitudinal compressive forces $P = 30$ t applied to the end



PROBLEM 7.27

sections with an eccentricity $e = 6{\cdot}5$ cm (see figure). Check the strength and the stability of the I-beam, if

$$[\sigma] = 1600 \text{ kg/cm}^2,$$
$$E = 2 \times 10^6 \text{ kg/cm}^2,$$

and the factor of safety

$$k = 1{\cdot}6.$$

Solution. (1) Check of stability in the plane of the least rigidity. From the tables (see Appendix A) a No. 20b I-section has $F = 39{\cdot}5$ cm^2 and $i_{\min} = i_y = 2{\cdot}06$ cm. The slenderness ratio of the beam is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{1 \times 240}{2{\cdot}06} = 116{\cdot}5.$$

The factor reducing the allowable stress $\Phi = 0{\cdot}474$. The allowable force is

$$[P] = \Phi[\sigma]\, F = 0{\cdot}474 \times 1600 \times 39{\cdot}5 = 29{,}960 \text{ kg} \approx \text{P} = 30{,}000 \text{ kg}.$$

(2) Check of strength in the plane of the greatest rigidity. We carry out the check of strength using the formula

$$\sigma_{\text{eff}} = \frac{P}{F} + \frac{M_{\max}}{W_x}\left(1 + \frac{mk\dfrac{P}{P_{\text{crit}}}}{1 - k\dfrac{P}{P_{\text{crit}}}}\right) \leq [\sigma],$$

where $P_{\text{crit}} = (EJ_x\pi^2)/l^2$, k is the factor of safety, and m is a coefficient depending on the type of transverse loading. For a uniformly distributed load $m = 1{\cdot}028$; for a concentrated load at the mid-span $m = 0{\cdot}823$; for concentrated moments at the ends of the beam $m = 1{\cdot}234$.

In the given case $J_x = 2500$ cm^4, $W_x = 250$ cm, $M_{\max} = Pe = 30{,}000 \times 6{\cdot}5 = 195{,}000$ kgcm, $m = 1{\cdot}234$, and the ratio

$$\frac{P}{P_{\text{crit}}} = \frac{Pl^2}{EJ_x\pi^2} = \frac{30{,}000 \times 240}{2 \times 10^6 \times 2500 \times 9{\cdot}87} = 0{\cdot}035.$$

Therefore

$$\sigma_{\text{eff}} = \frac{30{,}000}{39{\cdot}5} + \frac{195{,}000}{250}\left[1 + \frac{1{\cdot}234 \times 1{\cdot}6 \times 0{\cdot}035}{1 - (1{\cdot}6 \times 0{\cdot}035)}\right]$$

$$= 760 + 837 = 1597 \text{ kg/cm}^2 \approx [\sigma] = 1600 \text{ kg/cm}^2.$$

7.28. A steel tube, with an external diameter of 60 mm and a wall thickness of 2 mm, is compressed by a force applied with an eccentricity of 2·5 mm. The length of the tube is 175 cm, its ends being pinned. A test of a short piece of the tube showed that the yield point of the material is 6600 kg/cm². Determine the allowable value of the compressive load, if the factor of safety k is 3.

Answer: 2·9 t.

7.29. A wooden post 2 m in length, one end of which is clamped, and the other free of constraints, has a square cross-section of 10×10 cm². The line of action of a longitudinal compressive load passes through a point lying on a diagonal of the square, at half the distance between the centre and one corner of the square. Determine the value of a load giving rise to a maximum compressive stress of 300 kg/cm².

Answer: 3485 kg.

7.30. A 75 mm × 75 mm × 10 mm angle 1 m long is compressed by a force of 10 t. The rod is initially straight; its ends are pinned. The eccentricity of the compressive force in the plane of the least rigidity is 3 mm. Determine the value of the deflection of the angle, and find the value of the maximum and minimum compressive stresses in the mid-section (with respect to the length) of the bar.

Answer: 0·74 mm; 1064 kg/cm²; 322 kg/cm².

7.31. What will be the deflection, and the value of the maximum compressive stress, if instead of being straight, as in the preceding problem, the axis of the angle has an initial curvature, with a sag of 2 mm? Assume that the influence of the eccentricity and that of the initial curvature can be added.

Answer: 1·24 mm (in addition to the initial deflection of 2 mm); 1300 kg/cm².

7.32. An I-beam, with a span of 6 m, carries a uniformly distributed load of 1·35 t/m, and is compressed by a longitudinal force of 60 t. In the direction perpendicular to the plane of the web of the I-section the span of the beam is divided into two halves by bracings. Select the cross-section of the beam for an allowable stress of 1600 kg/cm².

Answer: I-section No. 36a.

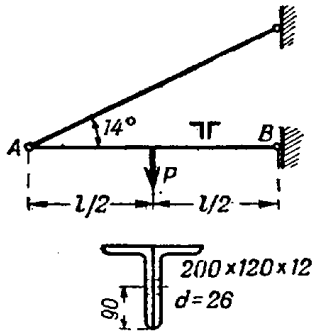
7.33. A steel rod, 25 mm in diameter and 2 m in length, with pinned ends, is loaded by axial compressive forces of 400 kg, and a tranverse force of 4 kg at the mid-span. Determine the value of the deflection and the value of the maximum stress in the middle section (with respect to the length) of the rod (a) neglecting the weight of the rod, and (b) taking the weight of the rod into account.

Answer: (a) 3 mm and 290 kg/cm²; (b) 6·7 mm and 508 kg/cm².

7.34. The beam AB of a bracket, 5 m long, hinged at points A and B, carries a concentrated transverse load $P = 2{\cdot}9$ t applied to the middle of the span. The transverse section of the beam is made up of two 200 mm × 120 mm × 12 mm angles (see figure).

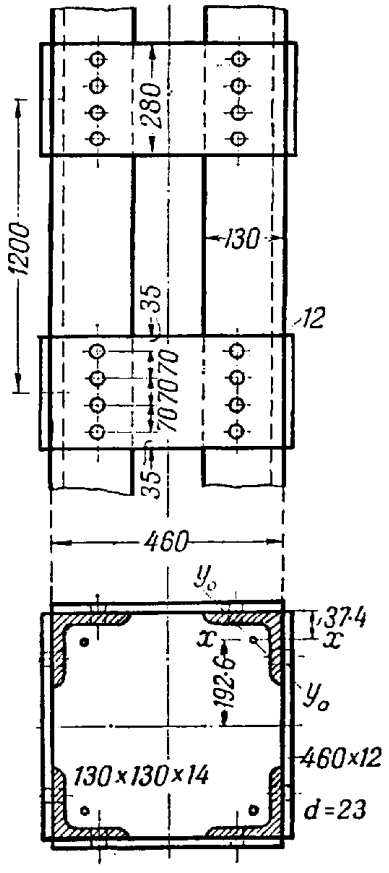
Check the strength and the stability of the beam, with the assumption that joint A cannot move in the direction perpendicular to the plane of the figure. The allowable stress $[\sigma]$ is 1600 kg/cm²; $k = 1{\cdot}6$.

Answer: From the condition of limiting stability $[P] = 28{\cdot}5$ t; $\sigma_{\text{eff}} = 1613$ kg/cm².



PROBLEM 7.34

7.35. Check the strength and the stability of a pin-ended column with a bracing in the form of plates (see figure), having a length of 12 m, and being loaded by longitudinal compressive forces $P(= 180\text{ t})$. The basic allowable stress $[\sigma]$ is 1600 kg/cm².



PROBLEM 7.35

Solution. (1) *Check of stability.* Since the area and the second moment of area of the cross-section of one 130 mm × 130 mm × 14 mm angle are respectively $F = 34{\cdot}7\text{ cm}^2$ and $J_x = 545\text{ cm}^4$, then for the whole cross-section $F = 4 \times 34{\cdot}7 = 138{\cdot}8\text{ cm}^2$, $J = 4[545 + 34{\cdot}7(23 - 3{\cdot}74)^2] = 53{,}670\text{ cm}^4$, and the radius of gyration

$$i = \sqrt{\frac{J}{F}} = \sqrt{\frac{53{,}670}{138{\cdot}8}} = 19{\cdot}66\text{ cm}.$$

The length factor for a composite column with a bracing of plates is determined by the formula

$$\mu = \sqrt{1 + \frac{\pi^2 J a^2}{24 J_1 l^2}\left(1 + \frac{2bJ_1}{aJ_P}\right)},$$

where J is the second moment of area of the whole cross-section of the column, J_1 is the second moment of area of one half of the cross-section (consisting in the given case of two angles) about its own centroidal axis, J_p is the second moment of area of the cross-section of one plate about its centroidal axis perpendicular to the plane of the plate, a is the distance between the centre lines of the plates measured along the length of the column, b is the distance between the centroids of the halves of the cross-section. In the given case $l = 1200$ cm, $a = 120$ cm, $b = 46 - (2 \times 3{\cdot}74) = 38{\cdot}52$ cm, $J_1 = 2J = 2 \times 545 = 1090$ cm^4 and $J_p = (1{\cdot}2 \times 28^3)/12 = 2195$ cm^4.

Thus

$$\mu = \sqrt{1 + \frac{9{\cdot}87 \times 53{,}670 \times 120^2}{24 \times 1090 \times 1200^2}\left(1 + \frac{2 \times 38{\cdot}52 \times 1090}{120 \times 2195}\right)} = 1{\cdot}1.$$

The effective slenderness ratio of the column is

$$\lambda = \frac{\mu l}{i} = \frac{1{\cdot}1 \times 1200}{19{\cdot}66} = 67{\cdot}1.$$

The factor reducing the allowable stress $\Phi = 0{\cdot}824$. The value of the allowable load is

$$[P] = 0{\cdot}824 \times 1600 \times 138{\cdot}8 = 183{,}000 \text{ kg} > P = 180{,}000 \text{ kg}.$$

(2) *Check of strength.* Taking into account that the cross-section of the column is weakened by the rivet holes, we have

$$F_{\text{net}} = F_{\text{gross}} - 8d\delta_a = 138.8 - (8 \times 2{\cdot}3 \times 1{\cdot}4) = 113 \text{ cm}^2.$$

The compressive stress in the cross-section thus weakened is

$$\sigma = \frac{P}{F_{\text{net}}} = \frac{180{,}000}{113} = 1593 \text{ kg/cm}^2 < [\sigma] = 1600 \text{ kg/cm}^2.$$

The strength and the stability of the column is fully guaranteed.

(3) *Check of stability of one leg of the column within the limits of one panel length.* The minimum second moment of area, and the radius of gyration of the cross-section of one angle are respectively $J_{y_0} = 223 \text{ cm}^4$ and $i_{\min} = i_{y_0} = 2{\cdot}54 \text{ cm}$. Assuming the end fixing of one leg at the plate as pin-jointed, and the length of the leg equal to the distance between the centre lines of the end rivets, we have $l_1 = 120 - (3 \times 7) = 99 \text{ cm}$, and the slenderness ratio of the leg

$$\lambda_1 = \frac{\mu l_1}{i_{\min}} = 39 < 69{\cdot}4.$$

Since $\lambda_1 < l$, the determination of the value of the allowable load on the leg cannot be carried out.

7.36. Check the strength and the stability of a composite column 4 m in length having a bracing of spacers and diagonal members (see figure). The ends of the column are pinned. The load P is 100 t, $[\sigma]$ is 1600 kg/cm^2.

Solution. (1) *Check of stability.* The area and the second moment of area of the cross-section of one channel are respectively $F = 39 \text{ cm}^2$ and $J_y = 194{\cdot}1 \text{ cm}^4$. For the whole cross-section $F = 2 \times 39 = 78 \text{ cm}^2$ and

$$J_{y_0} = 2\left[194{\cdot}1 + 39\left(\frac{24}{2} - 8 + 2{\cdot}03\right)^2\right] = 3224 \text{ cm}^4 < J_{x_0}$$

$$= 2 \times 3283 = 6566 \text{ cm}^4.$$

The minimum radius of gyration of the cross-section is

$$i_{\min} = i_{y_0} = \sqrt{\frac{J_{y_0}}{F}} = \sqrt{\frac{3224}{78}} = 6{\cdot}43 \text{ cm}.$$

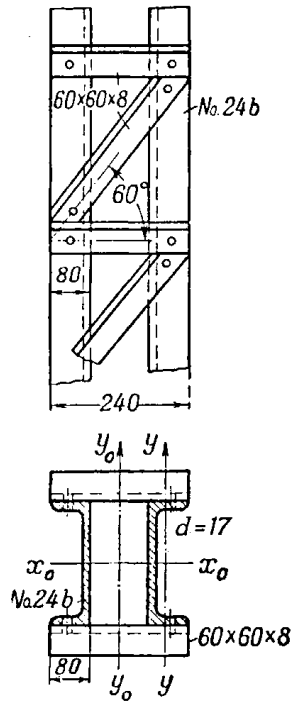
The length factor μ for a composite column with a bracing of spacers and diagonal members will be determined from the formula

$$\mu = \sqrt{1 + \frac{\pi^2 J}{2l^2}\left(\frac{1}{F_{sp}\tan\alpha} + \frac{1}{F_d \cos^2\alpha \sin\alpha}\right)}$$

where J is the second moment of area of the whole cross-section about the y_0 axis, F_{sp} and F_d are the cross-sectional areas of the spacer and the diagonal member respectively, α is the angle between the spacer and the diagonal member, and l is the length of the column. In the given case $J = J_{y_0}$, $F_{sp} = F_d = 9{\cdot}03$ cm^2, $\alpha = 60°$. Therefore

$$\mu = \sqrt{1 + \frac{9{\cdot}87 \times 3224}{2 \times 400^2}\left(\frac{1}{9{\cdot}03 \times \sqrt{3}} + \frac{1}{9{\cdot}03 \times \frac{1}{4} \times \frac{\sqrt{3}}{2}}\right)}$$

$$= 1{\cdot}028.$$



PROBLEM 7.36

The effective slenderness ratio of the column $\lambda = \mu l/i_{\min} = (1{\cdot}028 \times 400)/6{\cdot}43 = 63{\cdot}9$. The factor reducing the allowable stress $\Phi = 0{\cdot}84$. The allowable load is

$$[P] = \Phi[\sigma]\ F = 0{\cdot}84 \times 1600 \times 78 = 104{,}800\ \text{kg} > P = 100{,}000\ \text{kg}$$

(2) *Check of strength.* The cross-sectional area of the column is reduced by four rivet holes 17 mm in diameter. We have

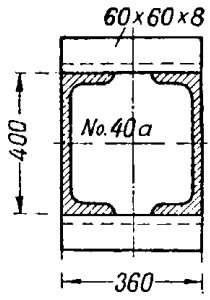
$$F_{\text{net}} = 78 - (4 \times 1{\cdot}7 \times 1{\cdot}2) = 69{\cdot}8\ \text{cm}^2.$$

The compressive stress in the weakened cross-section is

$$\sigma = \frac{P}{F_{\text{net}}} = \frac{100{,}000}{69{\cdot}8} = 1435\ \text{kg/cm}^2 < [\sigma] = 1600\ \text{kg/cm}^2.$$

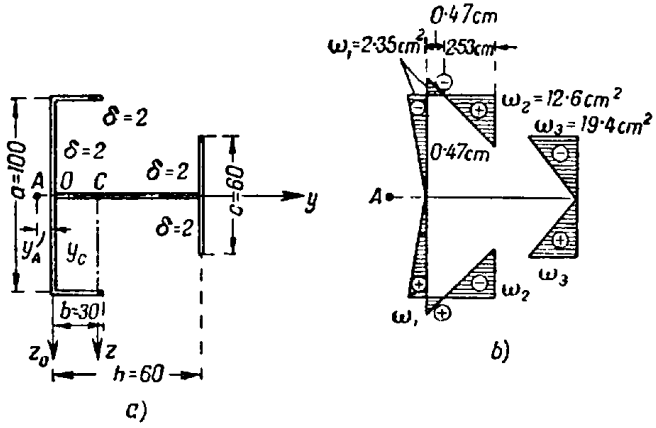
7.37. A composite column with a bracing of spacers and diagonal members (see figure) is pinned at the ends and has a length of 13 m. The spacers and the diagonal members are 60 mm × 60 mm × 8 mm angles welded to the flanges. The angle α between the spacers and the diagonal members is 60°. $[\sigma] = 1600\ \text{kg/cm}^2$. Determine the maximum allowable value of the compressive load.

Answer: 173 t.



PROBLEM 7.37

7.38. Determine the value of the critical force for a steel column having a cross-section shown in figure (a). The length of the column is 3 m; its ends are pinned. The compressive force is applied to the centroid of the cross-section.



PROBLEM 7.38

Solution. The geometrical characteristics of the cross-section:

The area of the cross-section $F = 5{\cdot}6$ cm².

The distance from the z_0 axis to the centroid C of the cross-section $y_C = 2{\cdot}25$ cm.

The second moments of area of the cross-section about the principal centroidal axes $J_y = 50{\cdot}2$ cm⁴, $J_z = 32{\cdot}9$ cm⁴.

The second moment of area of the cross-section for pure shear $J_t = 0{\cdot}0747$ cm⁴.

The distance from point O to the shear centre A of the cross-section (see figure (a)) $y_A = -0{\cdot}47$ cm.

The principal sectorial second moment of area of the cross-section (see figure (b)) $J_\omega = 208{\cdot}5$ cm⁶.

The distance from the principal z axis to the centre of shear of the cross-section

$$a_y = y_C + |y_A| = 2{\cdot}25 + 0{\cdot}47 = 2{\cdot}72 \text{ cm}.$$

The resultant characteristic of the cross-section

$$r^2 = \frac{J_y + J_z}{F} + a_y^2 = \frac{50{\cdot}2 + 32{\cdot}9}{5{\cdot}6} + 2{\cdot}72^2 = 22{\cdot}2 \text{ cm}^2.$$

The critical force according to Euler in the plane of maximum rigidity is

$$P_y = \frac{EJ_y\pi^2}{l^2} = \frac{2 \times 10^6 \times 50{\cdot}2 \times 3{\cdot}14^2}{300^2} = 11{,}000 \text{ kg}.$$

The critical force (according to Euler) in the plane of minimum rigidity is

$$P_z = \frac{EJ_z\pi^2}{l^2} = \frac{2 \times 10^6 \times 32{\cdot}9 \times 3{\cdot}14^2}{300^2} = 7210 \text{ kg}.$$

The force corresponding to the torsional effect is

$$P_\omega = \frac{1}{r^2}\left(\frac{EJ_\omega\pi^2}{l^2} + GJ_t\right) = \frac{1}{22{\cdot}2}\left[\frac{2 \times 10^6 \times 208{\cdot}5 \times 3{\cdot}14^2}{300^2} + (8 \times 10^5 \times 0{\cdot}0747)\right] = 4750 \text{ kg}.$$

The value of the critical force according to Vlasov is

$$P_3 = \frac{(P_y + P_\omega)r^2 - \sqrt{(P_y + P_\omega)^2 r^4 - 4P_y P_\omega r^2 (r^2 - a_y^2)}}{2(r^2 - a_y^2)}$$

$$= \frac{(11{,}000 + 4750)22{\cdot}2 - \sqrt{(11{,}000+4750)^2 22{\cdot}2^2 - [4 \times 11{,}000 \times 4750 \times 22{\cdot}2(22{\cdot}2 - 2{\cdot}72^2)]}}{2(22{\cdot}2 - 2{\cdot}72^2)}$$

$$= 3995 \text{ kg}.$$

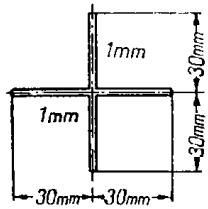
The buckling of the column takes place in a mode of flexure and torsion. The magnitude of the critical force according to Vlasov is 7210/3995 = 1·8 times smaller than the critical force according to Euler.

7.39. Determine the value of the critical force for a No. 8 channel having a length of 1·5 m. The ends of the channel are pinned; the compressive force is applied to the centroid of the cross-section. When solving the problem make use of the data on standard rolled sections (see Appendix A).

Answer: $P_{\text{crit}} = P_{\text{Euler}} = 14{,}550 \text{ kg} < P_3 = 55{,}500 \text{ kg}$ (the column buckles in a purely flexural mode).

7.40. Determine the value of the critical force for a steel column having the cross-section shown in the figure, with the assumption that the column has a length of (a) 1·5 m, and (b) 2 m. The ends

of the column are pinned; the compressive force is applied to the centroid of the cross-section.



PROBLEM 7.40

Answer: (a) $P_{\text{crit}} = P_{\omega} = 1065\,\text{kg} < P_{\text{Euler}} = 1580\,\text{kg}$ (the column buckles in a purely torsional mode); (b) $P_{\text{crit}} = P_{\text{Euler}} = 888\,\text{kg} < P_{\omega} = 1065\,\text{kg}$ (the column buckles in a purely flexural mode).

CHAPTER 8

METHOD OF ALLOWABLE LOADS

33. TENSION, COMPRESSION, AND TORSION

8.1. An *encastré*-ended vertical steel bar, with a cross-sectional area $F = 200\ \text{cm}^2$ and a length $l = 5\ \text{m}$, is loaded axially by a concentrated downward force P, applied at a distance $a = 2\ \text{m}$ from the lower end of the bar.

Determine by the method of allowable loads the magnitude of the safe load P, if the yield stress σ_y of the material of the bar is $2400\ \text{kg/cm}^2$. Take the factor of safety as equal to 2.

Solution. The collapse load will be the one for which the material yields both in the lower and upper parts of the bar.

As long as the material works within the elastic limits the force P is balanced by the upward reactions B and A at the upper and lower supports respectively. The equation of equilibrium has the form

$$A + B = P.$$

The equation of compatibility of deformations states that the elongation of the upper part of the bar is numerically equal to the contraction of the lower part, i.e.

$$\frac{Aa}{EF} = \frac{B(l-a)}{EF}.$$

Combining the equation of equilibrium with the equation of deformations, we find that

$$A = P\frac{l-a}{l} \quad \text{and} \quad B = P\frac{a}{l}.$$

Since $l - a > a$, the most intensely stressed part will be the lower one, being compressed by the forces A. At the instant when the stresses in it reach the yield point the reaction of the lower support will be $A_y = \sigma_y F$. The force transmitted to the upper part at this instant will be

$$B' = P - A_y = P - \sigma_y F.$$

During a further increase of the force P the reaction $A_y = \sigma_y F$ will remain constant, and reaction B alone will increase. The load-carrying capacity of the rod will be exhausted when the stresses in the upper part also reach the yield point, i.e. when reaction B is $B_y = \sigma_y F$.

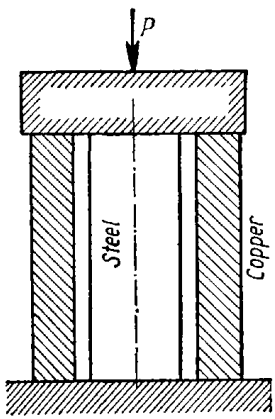
Thus

$$P_y = A_y + B_y = \sigma_y F + \sigma_y F = 2\sigma_y F.$$

With the factor of safety $k = 2$, the allowable load is $P_{max} = P_y/k = 2\sigma_y F/k = 2[\sigma]F$, where $[\sigma] = \sigma_y/k$ is the allowable stress. Thus

$$P_{max} = 2 \times \frac{2400}{2} \times 200 = 480{,}000 \text{ kg} = 480 \text{ t}.$$

8.2. A load P is transmitted by means of a rigid slab to a solid steel cylinder having a cross-sectional area of 15 cm², and a hollow copper cylinder having a cross-sectional area of 20 cm² (see figure). Determine the magnitude and the distribution of the load between



PROBLEM 8.2

the steel and the copper cylinder at the instant when the material yields in both cylinders. The yield point of steel is at $\sigma_{ys} = 2400\,\text{kg/cm}^2$; the yield point of copper is at $\sigma_{yc} = 1800\,\text{kg/cm}^2$.

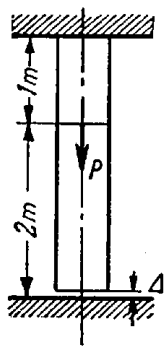
Answer: $P_s = P_c = 36$ t.

8.3. The cross-sectional area of the concrete in a short reinforced concrete column is 645 cm². The column is provided with four longitudinal steel rods arranged symmetrically, each with a cross-sectional area of 10 cm². The allowable stress is 80 kg/cm² for the concrete, and 1400 kg/cm² for the reinforcement. The factor of safety is to be taken the same for both. Determine by the method of allowable loads the value of a safe load.

Answer: 107·6 t.

8.4. A rod, with a cross-sectional area $F = 100$ cm², is clamped at the upper end and loaded as shown in the figure. Before loading between its lower end and a rigid plane there is a gap Δ of 0·02 mm. Find by the method of allowable loads the maximum safe value of the force P, if $[\sigma]$ is 1000 kg/cm².

Answer: 200 t.

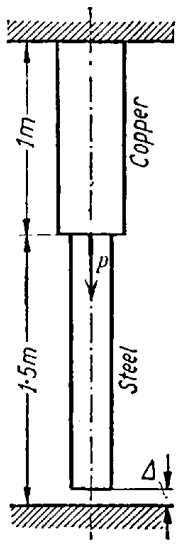


PROBLEM 8.4

8.5. The rod represented in the figure is fixed by its upper end. Before loading between its lower end and the rigid support there is a clearance Δ of 0·05 mm. The cross-sectional area of the upper part (made of copper) is 150 cm², whereas the cross-sectional area of the lower part (made of steel) is 50 cm². The allowable stresses are: $[\sigma_c] = 1000\,\text{kg/cm}^2$ for copper, $[\sigma_s] = 1400\,\text{kg/cm}^2$ for steel.

The factor of safety is to be taken the same for both. Determine by the method of allowable loads the maximum safe value of the force P.

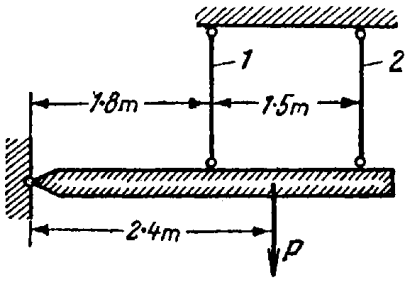
Answer: 220 t.



PROBLEM 8.5

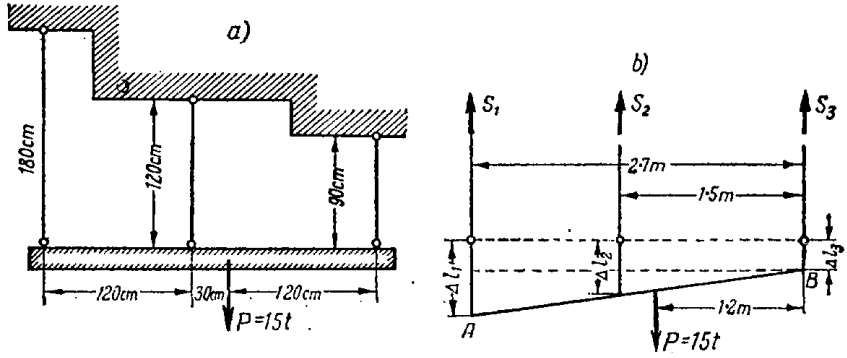
8.6. A rigid beam (see figure) is suspended from two rods. The area of the cross-section of the first rod is 10 cm², whereas that of the second is 15 cm². The yield stresses of the materials of the rods are respectively $\sigma_{y1} = 2600$ kg/cm² and $\sigma_{y2} = 1500$ kg/cm². Determine by the method of allowable loads the magnitude of the safe load P, if the factor of safety is 2.

Answer: 25.2 t.



PROBLEM 8.6

8.7. Three rods supporting a rigid beam (see figure (a)) are made of steel with an allowable stress $[\sigma]$ of 1600 kg/cm^2, and have identical cross-sections. Determine the required cross-section of the rods by the method of allowable loads.



PROBLEM 8.7

Solution. We shall consider first the distribution of the load between the rods in the elastic stage of deformation. We shall assume that all rods are in tension. Then adding the load and the vertical components of all forces in the rods, we have

$$S_1 + S_2 + S_3 - P = 0. \tag{a}$$

The sum of the moments of these forces about the hinge of the middle rod gives us the second equation of equilibrium

$$(S_1 \times 1{\cdot}2) + (P \times 0{\cdot}3) - (S_3 \times 1{\cdot}5) = 0. \tag{b}$$

Considering the deformations of the system shown in figure (b), we have

$$\frac{\Delta l_1 - \Delta l_3}{2{\cdot}7} = \frac{\Delta l_2 - \Delta l_3}{1{\cdot}5}.$$

We shall express the elongations in terms of the forces, and carry out the required simplifications. Then the equation of deformations assumes the following form

$$S_1 - 1{\cdot}2S_2 + 0{\cdot}4S_3 = 0. \tag{c}$$

Solving equations (a), (b) and (c) simultaneously, we obtain the following values of the forces, and the stresses in the elastic stage of deformation of the rods:

$$S_1 = 0{\cdot}253P \quad \text{and} \quad \sigma_1 = 0{\cdot}253\,\frac{P}{F},$$

$$S_2 = 0{\cdot}345P \quad \text{and} \quad \sigma_2 = 0{\cdot}345\,\frac{P}{F},$$

$$S_3 = 0{\cdot}403P \quad \text{and} \quad \sigma_3 = 0{\cdot}403\,\frac{P}{F}.$$

We notice that in the elastic stage of deformation rod 3 is the most stressed one. On increasing the force P the stress in it will reach the yield point first of all. For rod 3 the limiting load will have the value

$$S_3^{\lim} = \sigma_y F.$$

At this instant the stresses in rods 1 and 2 have not yet reached the yield point. During a further increase of the force, the stress, and consequently the force in rod 3 will remain unchanged, whereas the stresses in rods 1 and 2 will increase until the stress in rod 2 (the most severely stressed one of the remaining two) in its turn will just reach the yield point, and the force in it will become equal to

$$S_2^{\lim} = S_3^{\lim} = \sigma_y F.$$

This is the limiting state of the structure since any further increase of the load is impossible. Indeed, rods 2 and 3 will deform without any increase of the forces, and the rigid beam will begin to rotate about the hinge of rod 1.

We shall set up the equation of equilibrium for the instant when the structure reaches the limiting state, i.e. when the load reaches the limiting value $P^{\lim}$. We shall take moments about the hinge of the first rod:

$$(\sigma_y F \times 1{\cdot}2) + (\sigma_y F \times 2{\cdot}7) - (P^{\lim} \times 1{\cdot}5) = 0.$$

We shall divide this equation by the factor of safety, noting that $\frac{P^{\text{lim}}}{k} = P$ and $\frac{\sigma_y}{k} = [\sigma]$:

$$([\sigma]F \times 1{\cdot}2) + ([\sigma]F \times 2{\cdot}7) - (P \times 1{\cdot}5) = 0,$$

whence

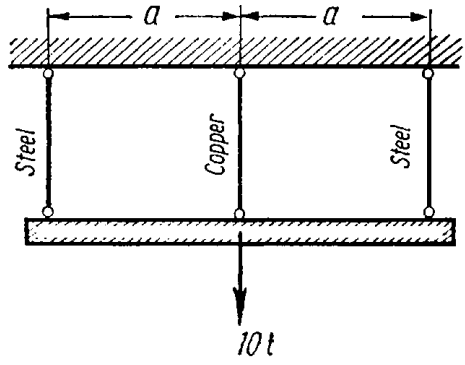
$$F = \frac{1{\cdot}5P}{[\sigma](1{\cdot}2 + 2{\cdot}7)} = \frac{1{\cdot}5 \times 15{,}000}{1600 \times 3{\cdot}9} = 3{\cdot}61 \text{ cm}^2.$$

8.8. Determine by the method of allowable loads the required dimensions of a concrete column of square cross-section, reinforced by four steel rods, the cross-sectional area of which constitutes 1% of the cross-sectional area of the column. The allowable stress for the concrete is 60 kg/cm²; for the steel it is 1200 kg/cm². The column carries a load of 100 t. What must be the side of the cross section of the column, and the diameter of the rods? Take the factors of safety for both the concrete and the reinforcement to be equal.

Answer: $a = 37$ cm; $d = 21$ mm.

8.9. A rigid beam (see figure) is suspended from three rods. The cross-sectional area of both steel rods is the same, whereas the cross-sectional area of the copper rod is $1\frac{1}{2}$ times greater. Assuming the yield point of steel $\sigma_{ys} = 2400$ kg/cm², that of copper $\sigma_{yc} = 1800$ kg/cm², and the factor of safety $k = 1{\cdot}5$, determine by the method of allowable loads the required cross-sectional areas of the rods.

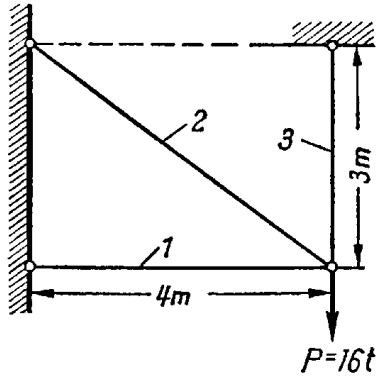
Answer: $F_S = 2$ cm²; $F_C = 3$ cm².



PROBLEM 8.9

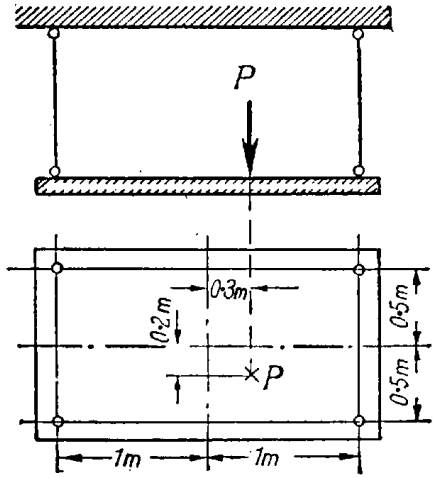
8.10. In the structure shown rod 1 is of aluminium with $[\sigma_a] = 800$ kg/cm², rod 2 is of copper with $[\sigma_c] = 600$ kg/cm², and rod 3 is of steel with $[\sigma_s] = 1200$ kg/cm². The cross-sectional areas of rods 1 and 2 are equal, whereas that of rod 3 is half that of rod 1. Determine the cross-sectional areas of the rods using the method of allowable loads.

Answer: $F_1 = F_2 = 16{\cdot}66$ cm²; $F_3 = 8{\cdot}33$ cm².



PROBLEM 8.10

8.11. A rigid rectangular slab is suspended from four rods of identical cross-section, length and material, arranged at its corners as shown in the figure. The allowable stress $[\sigma]$ of the material of



PROBLEM 8.11

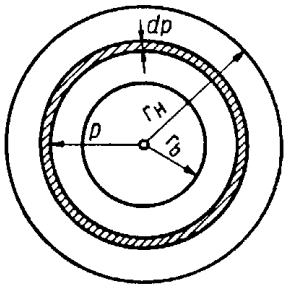
the hangers is 1600 kg/cm²; the cross-sectional area of the hangers is 4 cm² each. Determine by the method of allowable loads the safe value of the force P.

Answer: 18·3 t.

8.12. A rod of solid circular cross-section is subjected to a torque $M_t = 2090$ kgm. Determine by the method of allowable loads the required radius of the cross-section of the rod, if the allowable stress $[\tau]$ is 800 kg/cm².

Answer: 5 cm.

8.13. A rod of annular cross-section is to have a wall thickness equal to 0·1 times its external diameter. Determine by the method of allowable loads the external diameter of this rod, if it is subjected to a torque $M_t = 0{\cdot}9$ tm. The material yields at a stress $\tau_y = 1400$ kg/cm², and the factor of safety k is taken to be 2.



PROBLEM 8.13

Solution. According to the method of allowable loads, the limiting, unsafe load will be the one for which the stresses in the whole cross-section reach the yield point.

We shall consider an infinitely thin ring of the cross-section (see figure) having a thickness $\mathrm{d}\varrho$ and a radius ϱ. Its area is

$$\mathrm{d}F = 2\pi\varrho\,\mathrm{d}\varrho.$$

When the stresses reach the value σ_y, the moment of the force due to these stresses within the area of the ring, with respect to the centroid of the cross-section, is

$$\mathrm{d}M_y = 2\pi\varrho\,\mathrm{d}\varrho\,\tau_y\varrho.$$

For the whole cross-section

$$M_y = \int_{r_i}^{r_0} 2\pi\tau_y \varrho^2 \, d\varrho = \frac{2}{3}\pi\tau_y (r_0^3 - r_i^3) = \frac{2}{3}\pi r_0^3 \tau_y \left[1 - \left(\frac{r_i}{r_0}\right)^3\right].$$

We shall denote the ratio r_i/r_0 by α, and divide the value of this moment by the factor of safety k

$$\frac{M_y}{k} = \frac{2}{3}\pi r_0^3 \frac{\tau_y}{k}(1 - \alpha^3).$$

With the factor of safety equal to k, the quantity M_y/k represents the maximum safe value of the torque

$$M_t \leq \frac{2}{3}\pi r_0^3 \frac{\tau_y}{k}(1 - \alpha^3).$$

Hence we obtain the expression for the required dimension of the external radius of the cross-section

$$r_0 \geq \sqrt[3]{\frac{3kM_t}{2\pi\tau_y(1 - \alpha^3)}}.$$

In the given case $r_0 - r_i = (1/10)\ d_0 = r_0/5$, whence

$$\alpha = \frac{r_i}{r_0} = 0{\cdot}8.$$

Then

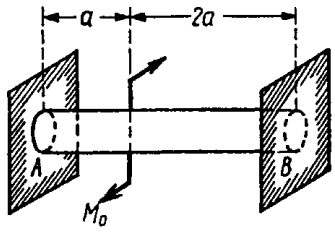
$$r_0 \geq \sqrt[3]{\frac{3 \times 2 \times 0{\cdot}9 \times 10^5}{2 \times 3{\cdot}14 \times 1400(1 - 0{\cdot}8^3)}} = 5 \text{ cm}.$$

and, consequently $d_0 = 2r_0 = 2 \times 5 = 10$ cm.

8.14. Using the method of allowable loads, determine the factor of safety for a rod of annular cross-section having an outside diameter d_0 of 12 cm and an inside diameter d, of 9 cm. It is subjected to a torque M_t of 4 t m. The yield point in shear τ_y of the material of the rod is 1050 kg/cm^2.

Answer: 2·5.

8.15. Using the method of allowable loads, determine the diameter of a solid bar clamped at both ends as shown in the figure, and subjected to a torque $M_0 = 3{\cdot}14$ tm, if the allowable stress $[\sigma]$ is 1200 kg/cm². The material of the bar is ductile; apply, therefore, the 3rd theory of strength.



PROBLEM 8.15

Solution. During the increase of the applied torque, as long as in any part of the bar the yield point is not reached by the whole cross-section, the left-hand reactive moment will be greater than the right-hand one. Due to this the cross-sections of the left-hand portion will be the first to yield fully, after which the left-hand reactive moment will cease to increase. The right-hand reactive moment will continue to increase as long as cross-sections of the right-hand portion are not fully involved in the plastic deformation. The plastic deformation of both parts of the bar determines the impossibility of a further increase of the load, establishing the so-called limiting state of the structure. In this state, in view of the uniform cross-section of the bar, the reactive moments are equal to each other.

It is obvious from the condition of equilibrium that each of the reactive moments is

$$M_A = M_B = \frac{M_0}{2}.$$

Thus the required torque is equal to

$$M_t = \frac{M_0}{2} = \frac{3{\cdot}14}{2}\, 10^5 \text{ kgcm}.$$

According to the third theory of strength the allowable shear stress $[\tau] = 0{\cdot}5\,[\sigma]$. In the given case

$$[\tau] = 0{\cdot}5 \times 1200 = 600 \text{ kg/cm}^2.$$

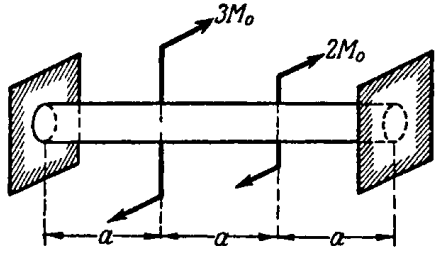
The required diameter of the cross-section is determined by the formula

$$d = 2\sqrt[3]{\frac{3}{2}\frac{M_t}{\pi[\tau]}} = \sqrt[3]{\frac{12M_t}{\pi[\tau]}} = \sqrt[3]{\frac{12 \times 3{\cdot}14 \times 10^5}{2 \times 3.14 \times 600}} = 10 \text{ cm}.$$

8.16. Using the method of allowable loads, determine the safe value, M, for a bar of solid circular cross-section, with a diameter $d = 7$ cm, clamped at both ends and loaded as shown in the figure, if the yield stress τ_y of the material of the bar is 2500 kg/cm², and the factor of safety, k, is assumed to be 2. Apply the fourth theory of strength.

Instruction. The ratio of the yield points of shear stress and normal stress is determined by the same considerations as the ratio of allowable stresses.

Answer: 0·27 tm.



PROBLEM 8.16

34. STATICALLY DETERMINATE BEAMS

8.17. Using the method of allowable loads, determine the gain factor of the calculated load-carrying capacity of a No. 30a I-beam, and find the value of a bending moment which may be applied at the critical section of the beam, if $[\sigma]$ is 1400 kg/cm².

Solution. The value of the limiting bending moment is $M_y = 2S\sigma_y$, where S is the first moment of area of one half of the cross-section with respect to the neutral axis, σ_y is the yield stress. The allowable bending moment is

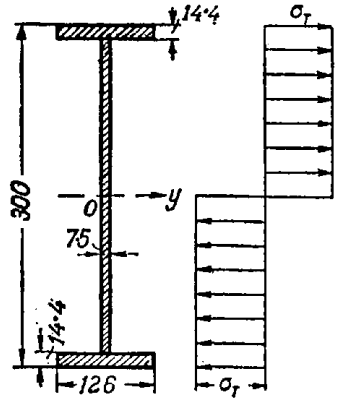
$$[M] = \frac{M_y}{k_y} = 2S\frac{\sigma_y}{k_y} = 2S[\sigma],$$

where $[\sigma]$ is the allowable stress. Designating $2S/W = n$, it is possible to write

$$[M] = [\sigma] \times W \times n.$$

The dimensions of the cross-section of a No. 30a beam are given in the figure. The first moment of area of one half of the cross-section is

$$S = (1{\cdot}44 \times 12{\cdot}6 \times 14{\cdot}28) + 0{\cdot}9\,\frac{13{\cdot}56^2}{2} = 341 \text{ cm}^3.$$



PROBLEM 8.17

With the given allowable stress, we obtain the value of the maximum bending moment

$$[M] = 2 \times 1400 \times 341 = 955{,}000 \text{ kg cm}.$$

Now the factor n, called the gain factor of the calculated load-carrying capacity of the beam, is

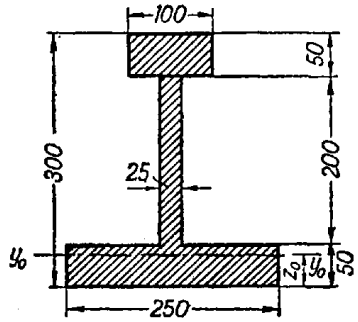
$$n = \frac{2S}{W} = \frac{2 \times 341}{597} = 1{\cdot}142.$$

8.18. Determine the limiting load-carrying capacity (q_y) of a simply supported No. 30a I-beam loaded by a uniformly distributed load q, if the span l of the beam is 5 m, and the yield stress σ_y of the steel is 2400 kg/cm².

Answer: $q_y = 5.35$ t/m.

8.19. Using the method of allowable loads, determine the load-carrying capacity of a steel I-beam shown in the figure, if the yield point σ_y of the material is 2400 kg/cm², and the factor of safety k_y is assumed to be 1·6. The span l of the beam is 4 m. A force P is applied at the mid-span.

How will the value of the allowable load be altered if the calculation is carried out on the basis of allowable stresses?



PROBLEM 8.19

Solution. Since in the plastic state of flexure the neutral axis divides the cross-sectional area into two equal portions, it must pass through the bottom flange (the area of the bottom flange is $5 \times 25 = 125$ cm², whereas that of the remaining portion of the cross-section is $(5 \times 10) + (2{\cdot}5 \times 20) = 100$ cm²).

The distance to the zero axis $y_0 - y_0$, denoted by z_0, will be from the condition of equality of the areas:

$$25z_0 = (5 \times 10) + (2{\cdot}5 \times 20) + (5 - z_0)25,$$

whence $z_0 = 4{\cdot}5$ cm.

The first moments of area about the neutral axis will be respectively

$$S_1 = (50 \times 23) + (50 \times 10{\cdot}5) + (25 \times 0{\cdot}5 \times 0{\cdot}25) = 1678 \text{ cm}^3$$

(for the upper part),

and

$$S_2 = 25 \times 4{\cdot}5 \times 2{\cdot}25 = 253 \text{ cm}^3 \text{ (for the lower part).}$$

Since the allowable stress $[\sigma] = \sigma_y/k_y = 2400/1{\cdot}6 = 1500$ kg/cm², then the allowable value of the bending moment is

$$[M] = (S_1 + S_2)[\sigma]$$
$$= (1678 + 253) \times 1500 = 2{,}900{,}000 \text{ kg cm} = 29 \text{ tm}.$$

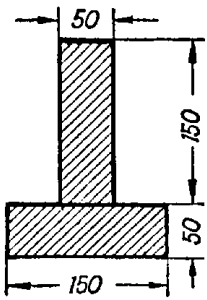
The allowable load is

$$[P] = \frac{4[M]}{l} = \frac{4 \times 29}{4} = 29 \text{ t}.$$

From the calculations based on allowable stresses $[P]_0 = 20$ t. Thus, using the method of calculation which takes into account plastic deformations it is possible to increase the calculated load-carrying capacity of the beam 29/20 = 1·45 times, i.e. by 45%.

8.20. Using the method of allowable loads, determine the load-carrying capacity of the T-beam shown in the figure, if the allowable stress $[\sigma]$ is 1600 kg/cm². The beam, having a span l of 2 m, is simply supported and loaded by a force P at the mid-span.

Answer: $P = 24$ t.



PROBLEM 8.20

8.21. Determine the limiting load-carrying capacity of a steel beam of hollow circular cross-section, simply supported and loaded by a continuous, uniformly distributed load q. The span of the beam is 6 m, the external diameter is 25 cm, the internal diameter is 20 cm, and the yield point of the steel is 2200 kg/cm². Find also the allowable load q, with the factor of safety $k_y = 1{\cdot}5$.

Answer: $q_y = 6{\cdot}2$ t/m; $[q] = 4{\cdot}1$ t/m.

8.22. Compare the load-carrying capacity of the beam considered in the preceding problem (which is calculated by the method of allowable loads) with the load-carrying capacity determined by the usual method of calculation.

Answer: $n = \dfrac{2S}{W} = 1{\cdot}35.$

8.23. Compare the value of factor n (which expresses the ratio $2S/W$) of a solid rectangular cross-section with that of a hollow one, if their respective dimensions are: (a) $b = 12$ cm, $h = 20$ cm; (b) $b_1 = 12$ cm, $b_2 = 10$ cm, $h_1 = 20$ cm and $h_2 = 12$ cm. The subscripts 1 and 2 refer to the external and internal dimensions of the hollow cross-section respectively.

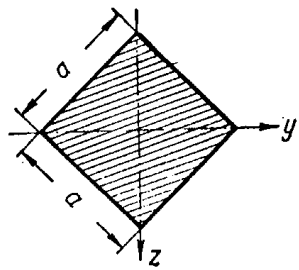
Answer: (a) $n = 1{\cdot}5$; (b) $n = 1{\cdot}28$.

8.24. Solve the preceding problem in the general form, assuming that

$$\frac{h_1}{h_2} = \alpha \quad \text{and} \quad \frac{b_1}{b_2} = \beta.$$

8.25. Find the ratio $2S/W$ for a square cross-section of a beam, having sides a, if the plane of flexure passes through one diagonal of the square.

Answer: $n = 2$.

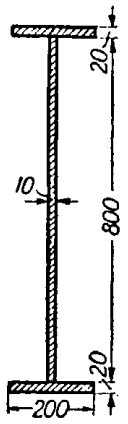


PROBLEM 8.25

8.26. Using the method of allowable loads, determine the limiting value of two concentrated forces P which may be safely applied, at equal distances a (= 2 m) from the respective supports, to a

fabricated beam having a span l (= 8 m). The cross-section of the beam is shown in the figure. Assume that the allowable stress $[\sigma]$ is 1600 kg/cm². The dimensions given are in mm.

Answer: $[P] = 39$ t.



PROBLEM 8.26

35. STATICALLY INDETERMINATE BEAMS

8.27. Using the method of allowable loads, determine the load-carrying capacity $[P_1]$ of the beam shown in figure (a), and compare it with the load-carrying capacity $[P_s]$ calculated on the basis of allowable stresses. The dimensions are: $a = 0{\cdot}6$ m; $b = 1{\cdot}3$ m; $c = 2{\cdot}0$ m. The allowable stress $[\sigma]$ is 1400 kg/cm²; the cross-section of the beam is I No. 22a.

Solution. Assuming all stresses remain elastic the bending moment diagram (with $P = 1$ t) is as shown in figure (b) (see Problem 5.112). Subsequently, on increasing the load, a plastic hinge is first of all formed at the point of application of force $2P$, and then at the *encastré* end. The diagram is shown in figure (c).

The bending moment at the point of application of force $2P$ may be expressed in the form

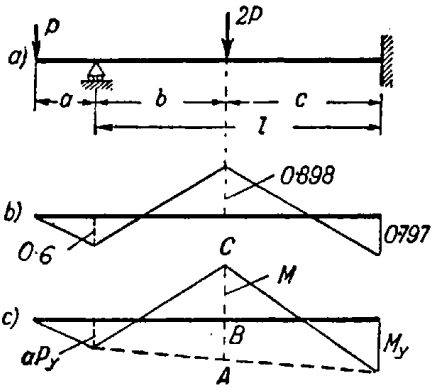
$$M_y = AC - AB = \frac{2P_y bc}{l} - \frac{aP_y c}{l} - \frac{M_y b}{l}.$$

The force P at the instant of failure is designated by P_y. It can be determined from the equation above

$$P_y = \frac{M_y(l+b)}{(2b-a)c}.$$

Bearing in mind that the collapse bending moment $M_y = 2S\sigma_y$, where S is the first moment of area of one half of the cross-section about the neutral axis, we have

$$P_y = \frac{2S\sigma_y(l+b)}{(2b-a)c}.$$



PROBLEM 8.27

The allowable load may be obtained by dividing the collapse load by the factor of safety k. If we assume that $\sigma_y/k = [\sigma]$, we have

$$[P_1] = \frac{2S[\sigma](l-b)}{(2b-a)c}.$$

For a No. 22a I-section, from the tables on standard sections, we have

$$\frac{J}{S} = 18{\cdot}9; \quad S = \frac{J}{18{\cdot}9} = \frac{3400}{18{\cdot}9} = 180 \text{ cm}^3.$$

Consequently,

$$[P_1] = \frac{2 \times 180 \times 1400 \times 460}{200 \times 200} = 5800 \text{ kg}.$$

In the calculation based on allowable stresses we can write (see figure (b))

$$[P_S] \times 89{\cdot}8 = W[\sigma],$$

whence

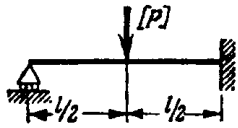
$$[P_S] = \frac{W[\sigma]}{89{\cdot}8} = \frac{309 \times 1400}{89.8} = 4820 \text{ kg}.$$

The reduction, in comparison with $[P_1]$, is

$$\frac{5800 - 4820}{5800} \times 100 = 16{\cdot}9\%.$$

8.28. Using the method of allowable loads, determine the load-carrying capacity of the beam shown in the figure. $l = 2$ m; $[\sigma] = 1600$ kg/cm^2; the cross-section is I No. 16.

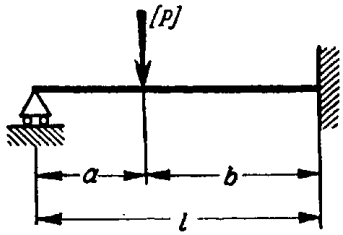
Answer: $[P] = 7850$ kg.



PROBLEM 8.28

8.29. For what position of the force $[P]$ will its value be the least, when determined by the method of allowable loads?

Answer: $a = 0{\cdot}414\,l$.

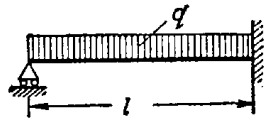


PROBLEM 8.29

8.30. Determine $[q]$, by using the method of allowable loads.

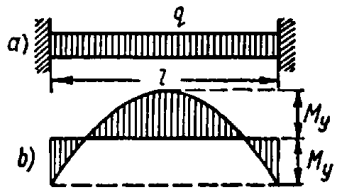
Instruction. To solve the problem it will be necessary to equate to M_y the maximum bending moment in the span, and that at the constraint.

Answer: $[q] = \dfrac{23{\cdot}26S[\sigma]}{l^2}$.



PROBLEM 8.30

8.31. Select from the standard profiles a cross-section for the beam shown in the figure. $[\sigma] = 1400$ kg/cm²; $l = 4$ m; $q = 5$ t/m. By an arrangement of the end fixings it is ensured that the horizontal components of the reactions at the supports are zero.



PROBLEM 8.31

Solution. Because of the symmetry of the system the bending moment diagram, at the instant when the plastic hinges are formed, is as shown in figure (b). We have

$$M_y = \frac{ql^2}{8} - M_y;$$

hence

$$M_y = \frac{ql^2}{16},$$

or, since

$$M_y = \sigma_y \times 2S, \quad \sigma_y \times 2S = \frac{q_y l^2}{16}.$$

Dividing both sides of the last equation by the factor of safety k, we obtain

$$[\sigma]2S = \frac{ql^2}{16},$$

whence

$$S = \frac{ql^2}{32[\sigma]} = \frac{50 \times 400}{32 \times 1400} = 178 \text{ cm}^3.$$

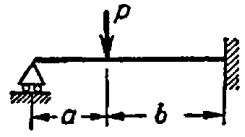
The ratio $J/S = \alpha$ is given in the tables on standard profiles. Consequently,

$$S = \frac{J}{\alpha}.$$

The selection of S should be made by a trial and error procedure. For a No. 22a I-section $J = 3400$ cm^4, $\alpha = 18{\cdot}9$ cm. Thus, $S = 3400/18{\cdot}9 = 180$ cm^3, which satisfies the required value $S = 178$ cm^3.

8.32. Applying the method of allowable loads, choose a rectangular cross-section, with a ratio of depth to width $h/c = 1{\cdot}5$, for the steel beam shown in the figure. $P = 1000$ kg; $[\sigma] = 1500$ kg/cm^2; $a = 30$ cm; $b = 70$ cm.

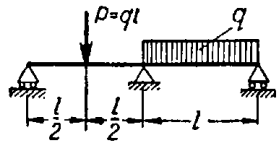
Answer: $2{\cdot}68 \times 4{\cdot}02$ cm^2.



PROBLEM 8.32

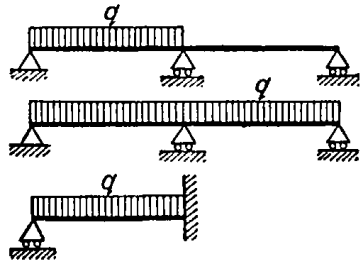
8.33. Using the method of allowable loads, choose a circular cross-section for the beam shown in the figure. $q = 500$ kg/m; $l = 1$ m; $[\sigma] = 1400$ kg/cm^2.

Answer: $d = 3{\cdot}1$ cm.



PROBLEM 8.33

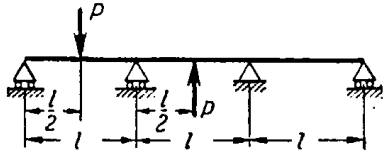
8.34. Three beams of the same material and cross-sectional area are loaded as shown in the figure. Which one of these beams will fail first, if the loads q increase simultaneously?



PROBLEM 8.34

8.35. Using the method of allowable loads, determine the factor of safety for the beam shown in the figure, assuming that the cross-section of the beam is I No. 20 a. $l = 4$ m; $P = 3$ t; $\sigma_y = 2600$ kg/cm^2.

Answer: 2·98.



PROBLEM 8.35

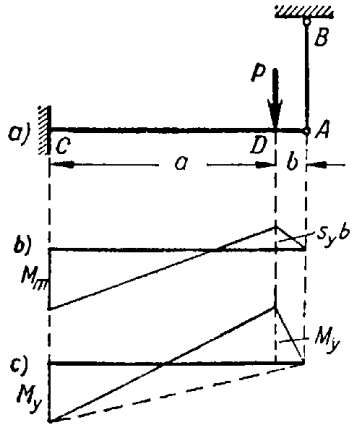
8.36. A No. 30 b I-beam is clamped at its left-hand end. The right-hand end is suspended from a rod having pin-joints at points A and B (see figure). Determine the value of the collapse load P_y, if $a = 8$ m, $b = 1$ m; the yield point (2400 kg/cm^2) is the same for beam AC and rod AB. The cross-sectional area F of the rod AB is 2·5 cm^2.

Solution. The collapse may occur according to this or that mode, depending on the ratio of dimensions of the cross-sections of rod AB and beam AC. If at the instant of collapse rod AB yields, and a plastic hinge is formed at the clamped end, the bending moment diagram has the form shown in figure (b). If rod AB has a large cross-section it cannot yield, and the collapse will be due to the formation of plastic hinges at points C and D. The bending moment

diagram corresponding to this is shown in figure (c). To solve the problem it is necessary to consider both of these cases.

For the case of figure (b) we have

$$M_y = P_y a - s_y l,$$



PROBLEM 8.36

where s_y is the force in rod AB at the onset of yield. Putting

$$M_y = 2S\sigma_y; \quad s_y = F\sigma_y,$$

we shall obtain

$$P_y = \frac{2S\sigma_y}{a} + \frac{F\sigma_y l}{a}.$$

From the table of standard sections (see Appendix A) we have

$$S = \frac{9400}{25{\cdot}4} = 370 \text{ cm}^3.$$

Consequently,

$$P_y = \frac{2 \times 370 \times 2400}{800} + \frac{2{\cdot}5 \times 2400 \times 900}{800} = 8970 \text{ kg} = 8{\cdot}97 \text{ t}.$$

For the case of figure (c) we have

$$M_y = \frac{P_y a b}{l} - M_y \frac{b}{l}.$$

Hence

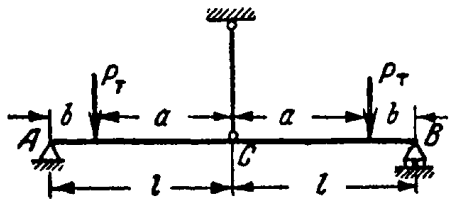
$$P_y = \frac{M_y\left(1 + \frac{b}{l}\right) l}{ab} = \frac{2 \times 370 \times 2400\left(1 + \frac{1}{9}\right) \times 900}{800 \times 100}$$

$$= 22{,}200 \text{ kg} = 22{\cdot}2 \text{ t}.$$

The collapse load will be obviously the smaller one of these loads, i.e. 8·97 t.

8.37. The beam AB is supported by its ends on rigid supports, and at its mid-point by a hinged rod. Determine the collapse load (P_y), if the yield point σ_y^r of the material of the rod is 2400 kg/cm², and that of the beam σ_y^b is 2800 kg/cm². $l = 6$ m; $b = 2$ m; the cross-section of the beam is I No. 18; the cross-sectional area F of the rod is 3 cm².

Answer: 6040 kg.



PROBLEM 8.37

CHAPTER 9

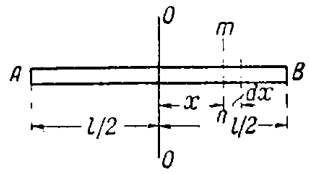
DYNAMIC AND CONTINUOUS LOADING

36. INERTIA FORCES

9.1. Determine the diameter of a 60 m long cable lifting, with a constant acceleration, a load weighing 5 t. In the first three seconds the load is lifted 9 m. The specific weight γ of the material of the cable is 7 g/cm^3; the allowable stress $[\sigma]$ is 600 kg/cm^2. Carry out the calculation (i) without the weight of the cable taken into account, and (ii) with the weight of the cable taken into account.

Answer: 3·58 cm; 3·74 cm.

9.2. A cast iron rod with a length l, of 1·8 m rotates with a constant angular speed about the vertical axis OO (see figure). Determine the limiting number of revolutions per minute for this rod, if the specific weight γ for cast iron is 7·4 g/cm^3, and the allowable tensile stress $[\sigma]$ is 400 kg/cm^2. Determine the elongation of the rod at 1000 rev/min, if E is 1·6 $\times$ 10^6 kg/cm^2.



PROBLEM 9.2

Solution. We shall take, at a distance x from the axis of rotation, an element of the rod with a length dx. The inertia force dP_a

acting on this element is the product of its mass and the radial acceleration, i.e.

$$\mathrm{d}P_{\mathrm{d}} = \frac{\gamma F\,\mathrm{d}x}{g}\,\omega^2 x,$$

where F is the area of the cross-section, g is the acceleration due to gravity, and ω is the angular speed of the rod.

The tensile stress in the section mn of the rod is

$$\sigma = \frac{P_{\mathrm{d}}}{F} = \frac{1}{F}\int_x^{\frac{l}{2}} \mathrm{d}P_{\mathrm{d}} = \int_x^{\frac{l}{2}} \frac{\gamma\,\omega^2 x\,\mathrm{d}x}{g} = \frac{\gamma\omega}{2g}\left(\frac{l^2}{4} - x^2\right).$$

The maximum tensile stress in the rod will occur at the axis of rotation, i.e. at $x = 0$; therefore the condition of the limiting stress will be written as

$$\sigma_{\max} = \frac{\gamma\omega^2 l^2}{8g} \leq [\sigma].$$

Hence the limiting value of the angular speed $[\omega]$ of the rod is

$$[\omega] = \sqrt{\frac{8g\,[\sigma]}{\gamma l^2}},$$

and the limiting number of revolutions per minute is

$$[n] = \frac{30[\omega]}{\pi} = \frac{30}{\pi}\sqrt{\frac{8g\,[\sigma]}{\gamma l^2}} = \frac{30}{3{\cdot}14}\sqrt{\frac{8 \times 981 \times 400}{0{\cdot}0074 \times 180^2}}$$

$$= 1090 \text{ rev/min}.$$

The elongation of the rod over a length $\mathrm{d}x$ is

$$\mathrm{d}\Delta l = \frac{\sigma\mathrm{d}x}{E} = \frac{\gamma\omega^2}{2gE}\left(\frac{l^2}{4} - x^2\right)\mathrm{d}x,$$

and the elongation of the whole rod is

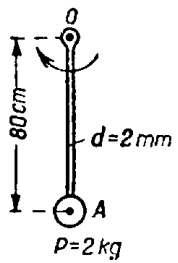
$$\Delta l = z\int_0^{\frac{l}{2}} \mathrm{d}\Delta l = 2\int_0^{\frac{l}{2}} \frac{\gamma\omega^2}{2gE}\left(\frac{l^2}{4} - x^2\right)\mathrm{d}x = \frac{\gamma\omega^2 l^3}{12gE}$$

$$= \frac{\gamma\pi^2 n^2 l^3}{12 \times 30^2 gE} = \frac{0{\cdot}0074 \times 3{\cdot}14^2 \times 1000^2 \times 180^3}{12 \times 30^2 \times 981 \times 1{\cdot}6 \times 10^6}$$

$$= 0{\cdot}0251 \text{ cm} = 0{\cdot}251 \text{ mm}.$$

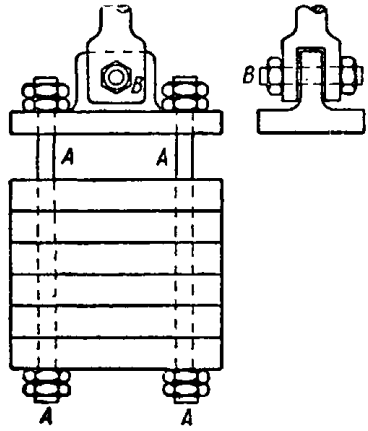
9.3. A steel wire OA, rotating steadily about the horizontal axis through point O, perpendicular to the plane of the drawing, carries a load of 2 kg at end A (see figure). At what rotational speed (in number of revolutions per minute) does the failure of the wire take place, if the ultimate strength of the steel is 8000 kg/cm²?

Answer: 375 rev/min.



PROBLEM 9.3

9.4. The balance weight of a lift (see figure) weighs 4200 kg. On braking of the ascending lift, the descending balance weight is



PROBLEM 9.4

subjected to an acceleration of 1·5 m/sec². Determine the diameter of the bolts A and B, if $[\sigma]$ is 320 kg/cm² and $[\tau]$ is 200 kg/cm² for the material of the bolts. The number of bolts A is two; the number of bolts B is one.

Answer: 31 mm; 39·3 mm.

9.5. What is the limiting rotational speed (in revolutions per minute about its axis) of a steel thin-walled cylinder, 30 cm in diameter, if the stress in it is not to exceed 800 kg/cm²?

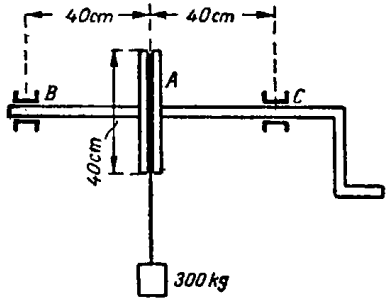
Answer: 6320 rev/min.

9.6. The maximum safe peripheral speed for cast iron flywheels is taken as 25 m/sec. Neglecting the spokes and taking the specific gravity as 7·4, determine the maximum tensile stress in the rim of the flywheel at this peripheral speed.

Answer: 47·1 kg/cm².

9.7. A load of 300 kg is raised with a uniform acceleration of 0·5 m/sec² by means of a cable (1 cm in diameter) which is wound on to a pulley 40 cm in diameter (see figure). Find the maximum effective stress in the cable, and shaft BC 5 cm in diameter, in accordance with the third theory of strength. Neglect the masses of the cable, pulley, and shaft.

Answer: σ_r (cable) = 401 kg/cm², σ_r (shaft) = 727 kg/cm².



PROBLEM 9.7

9.8. A load $P = 4$ t is raised with a uniform acceleration by means of a cable which is wound on a drum having a diameter D of 120 cm. The drum is mounted on a shaft which is driven by a motor.

The weight Q of the drum is 400 kg; its radius of gyration i is 60 cm. The acceleration, a, of the load is 5 m/sec^2. Determine the cross-sectional area of the cable, and the diameter of the shaft on which the drum is mounted, if the allowable stress $[\sigma]$ for the cable is 800 kg/cm^2, and $[\tau]$ for the shaft is 500 kg/cm^2.

Solution. The cable will be subjected to the action of a static force equal to load P, combined with the inertia force P_d, due to the load being lifted

$$P + P_d = P + \frac{Pa}{g} = P\left(1 + \frac{a}{g}\right) = 4000\left(1 + \frac{5}{9{\cdot}81}\right) = 6039 \text{ kg}.$$

The required cross-sectional area of the cable

$$F \geq \frac{P + P_d}{[\sigma]} = \frac{6039}{800} = 7{\cdot}55 \text{ cm}^2.$$

To determine the diameter of the shaft it will be necessary to find the twisting moment M_K. It is made up of two components. The first component (M'_K) is caused by the force in the cable, and is equal to

$$M'_K = (P + P_d)\frac{D}{2} = 6039 \times \frac{120}{2} = 362{,}300 \text{ kg cm}.$$

The second component (M''_K) is due to the inertia of the drum, and is equal to

$$M''_K = J_0 \varepsilon,$$

where ε is the angular acceleration, and J_0 the moment of inertia of the drum. The angular acceleration is equal to the tangential acceleration, $W_t = a$, of points on the circumference of the drum divided by the radius of the drum, i.e.

$$\varepsilon = \frac{2a}{D} = \frac{2 \times 500}{120} = \frac{25}{3} \text{ sec}^{-2}.$$

The moment of inertia of the drum

$$J_0 = \frac{Qi^2}{g} = \frac{400 \times 60^2}{981} = 1468 \text{ kg cm sec}^2.$$

Thus

$$M''_K = 1468 \times \frac{25}{3} = 12{,}200 \text{ kg cm},$$

and the total value of the twisting moment

$$M_K = M'_K + M''_K = 362{,}300 + 2{,}200 = 374{,}500 \text{ kg cm}.$$

The required diameter of the shaft

$$d \geq \sqrt[3]{\frac{16 M_K}{\pi[\tau]}} = \sqrt[3]{\frac{16 \times 374{,}500}{3{\cdot}14 \times 500}} = 15{\cdot}6 \text{ cm}.$$

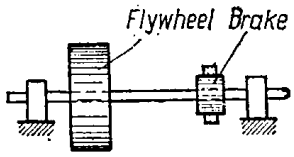
9.9. The rotational speed of a flywheel changes uniformly from 300 to 290 rev/min in 0·1 sec. The rim of the flywheel weighs 1·2 t; its radius of gyration is 50 cm. Determine the magnitude of the torque in the shaft on to which the flywheel is mounted, due to this change in the speed.

Answer: 320 kgm.

9.10. A leather belt 15 cm wide and 6 mm thick is slung over a pulley 1 m in diameter, and transmits 28 h.p. The pulley rotates with a constant speed of 450 rev/min. The weight of 1 cm³ of the belt material is 1 g. If the ratio of the forces in the tight and slack sides is 2·5, determine the stress in the belt (i) without the inertia forces in the belt taken into account, and (ii) with the inertia forces in the belt taken into account.

Answer: (i) 16·5 kg/cm²; (ii) 22·1 kg/cm².

9.11. A flywheel with a moment of inertia of 5000 kg cm sec² is mounted on to a shaft 100 mm in diameter (see figure). The rotational speed of the shaft is 300 rev/min. Suddenly the brake is



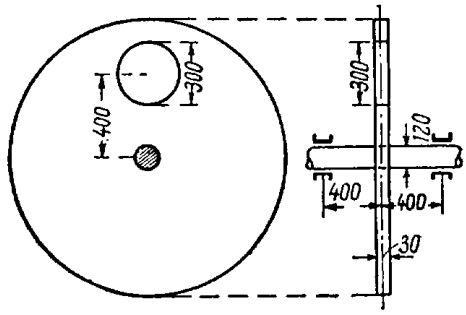
PROBLEM 9.11

applied, stopping the flywheel in 20 revolutions; the flywheel shaft is disengaged from the engine before the application of the brake. Determine the magnitude of the maximum shear stress in the shaft. Neglect the friction in the bearings.

Answer: 100 kg/cm².

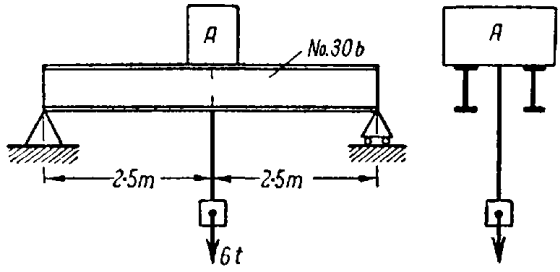
9.12. A solid steel disc 30 mm thick, rotating with a constant angular speed $\omega = 40$ per sec, is mounted on a shaft 120 mm in diameter. A circular hole 30 cm in diameter is made in the disc at a distance of 40 cm from the axis of the shaft (see figure). Find the increase in the maximum normal stress in the shaft, due to the presence of this hole.

Answer: 127 kg/cm².



PROBLEM 9.12

9.13. A lifting mechanism A weighing 2 t, and lifting a load of 6 t by means of a cable, is mounted on two No. 30b I-beams (see figure). If the load is lifted with a constant acceleration, and if it covers 2·5 m in the first second, determine the tension in the cable,

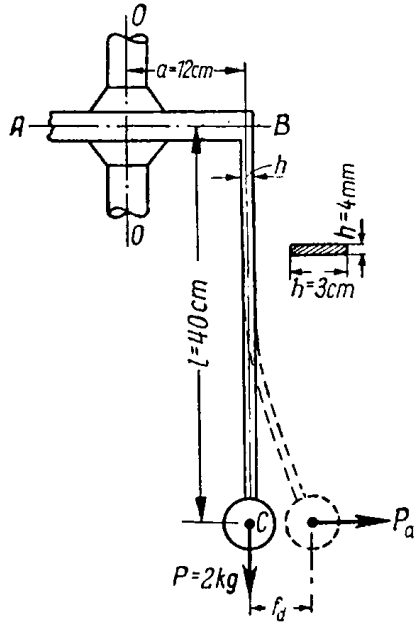


PROBLEM 9.13

and the value of the maximum normal stress in the beams. Take the weight of the beams and the lifting mechanism into account.

Answer: 9·06 t; 1130 kg/cm².

9.14. A governor rotates with a constant angular velocity about the OO axis (see figure). Find the limiting speed of the governor, and the deflection of the free end of the steel rod BC at this speed, if $[\sigma]$ is 1800 kg/cm². Assume the rod AB to be absolutely rigid; neglect the inertia forces in rod BC.



PROBLEM 9.14

Solution. During rotation the rod BC is being bent in the plane of the drawing by the centrifugal force P, the value of which depends on the angular velocity ω of the governor, and the distance r $(= a + f_d)$ of the load P from the axis of rotation, where $f_d = (P_d l^3)/3EJ$ is the deflection of the free end of rod BC. Thus,

$$P_d = \frac{P}{g}\omega^2 r = \frac{P}{g}\left(\frac{\pi n}{30}\right)^2 (a + f_d) = \frac{\pi^2 n^2}{900g} P\left(a + \frac{P_d l^3}{3EJ}\right),$$

whence

$$P_d = \frac{Pa}{\frac{900g}{\pi^2 n^2} - \frac{Pl^3}{3EJ}}. \qquad (a)$$

Since the maximum normal stress in the rod BC must not exceed the value of the allowable stress,

$$\max \sigma = \frac{M_{max}}{W} = \frac{P_d l}{W} \leq [\sigma],$$

and the maximum value of the force P_d is

$$P_d = \frac{[\sigma] W}{l}. \qquad (b)$$

Equating the expressions for P_d given by (a) and (b) we have

$$\frac{Pa}{\frac{900g}{\pi^2 n^2} - \frac{Pl^3}{3EJ}} = \frac{[\sigma] W}{l},$$

whence

$$n = \frac{30}{\pi}\sqrt{\frac{g}{Pl\left(\frac{a}{W[\sigma]} + \frac{l^2}{3EJ}\right)}} = \frac{30}{\pi}\sqrt{\frac{gbh^2}{6Pl\left(\frac{a}{[\sigma]} + \frac{2l^2}{3Eh}\right)}}$$

$$= \frac{30}{3{\cdot}14}\sqrt{\frac{981 \times 3 \times 0{\cdot}4^2}{6 \times 2 \times 40\left(\frac{12}{1800} + \frac{2 \times 40^2}{3 \times 2 \times 10^6 \times 0{\cdot}4}\right)}}$$

$$= 106 \text{ rev/min}.$$

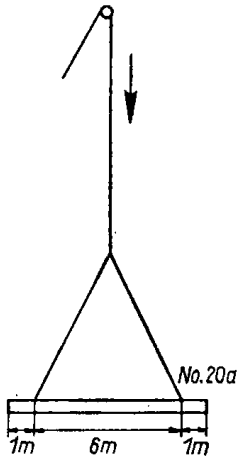
The deflection of the free end of the rod BC is

$$f_d = \frac{P_d l^3}{3EJ} = \frac{[\sigma] W l^2}{3EJ} = \frac{2[\sigma] l^2}{3Eh} = \frac{2 \times 1800 \times 40^2}{3 \times 2 \times 10^6 \times 0{\cdot}4} = 2{\cdot}4 \text{ cm}.$$

9.15. A channel No. 20a with a length of 8 m is lowered with a constant velocity of 1·8 m/sec by means of a cable (see figure).

Determine the maximum delfection of the channel, and the maximum normal stress in it due to flexure, if the velocity of descent in an interval of 0·2 sec decreases to 0·6 m/sec.

Answer: 2·08 cm; 603 kg/cm².

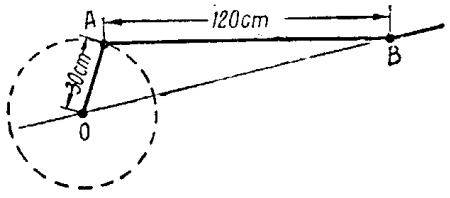


PROBLEM 9.15

9.16. A locomotive steel connecting rod of rectangular cross-section, with a depth of 120 mm and a width of 40 mm, has a length of 2·4 m. The radius of the crank is 30 cm. The compressive force in the connecting rod, when the rotational speed of the wheel is 200 rev/min, is 8·4 t. Determine the maximum normal stress in the connecting rod.

Answer: 580 kg/cm².

9.17. Determine the circular and square cross-sections necessary for the steel connecting rod AB (see figure), if the allowable stress



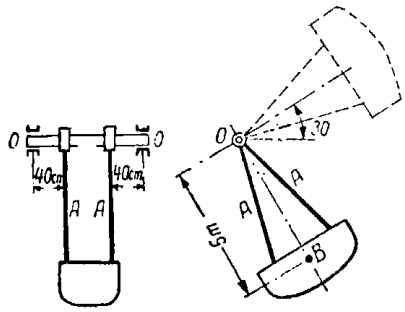
PROBLEM 9.17

$[\sigma]$ is 800 kg/cm^2. The crank OA makes 400 rev/min. Assume in the calculation that the centrifugal forces are perpendicular to the axis of the connecting rod, and vary along its length according to a linear law.

Answer: 38·6 $\approx$ 39 mm; 29 mm.

9.18. The weight of the cradle of a swing (see figure) when fully loaded is 300 kg. It is suspended by means of rods A from a steel shaft OO. Determine the required diameter of the shaft, assuming an allowable stress of 600 kg/cm^2, and considering the bending of the shaft due to inertia forces. Assume that the mass of the cradle is concentrated at point B, a distance of 5 m from the axis of rotation. The maximum angle of lift of the cradle above the horizontal is 30°.

Answer: 74 mm.



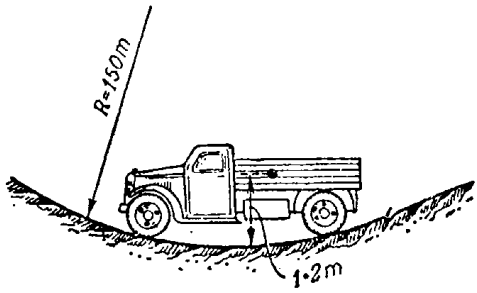
PROBLEM 9.18

9.19. A lorry moves with a speed of 45 km/hr along a curve, representing in the vertical plane an arc of a circle of radius (R) 150 m (see figure). The mass of the lorry and the load on it may be assumed to be concentrated at a point 1·2 m from the surface of the road. While the lorry is stationary a load equal to 1·5 t is transmitted to its rear wheels.

Determine the maximum normal stress in the leaf springs of the rear wheels while the lorry is stationary, and while it moves along the curve. Assume the springs to be beams of constant strength in flexure, simply supported and loaded at the mid-span. The span of

each spring is 80 cm, the width of each leaf is 6 cm, the thickness of the leaves is 9 mm, and the number of the leaves is 12.

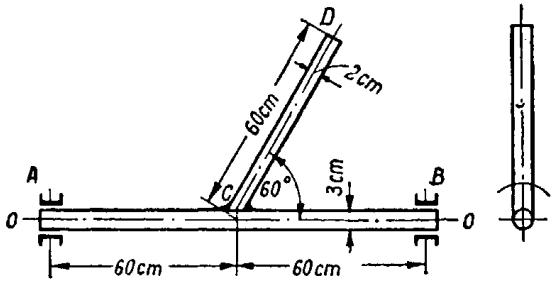
Answer: 1543 kg/cm²; 1708 kg/cm².



PROBLEM 9.19

9.20. A steel rod CD, 2 cm in diameter and 60 cm long, is rigidly fastened to the mid-span of a round steel rod AB 3 cm in diameter and 120 cm long (see figure). Rod AB rotates together with rod CD in bearings A and B about the OO axis with a constant angular velocity of 30 rad/sec. Determine the maximum normal stress in rods AB and CD.

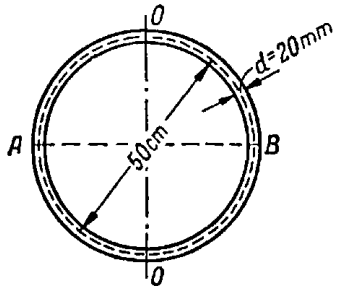
Answer: 529 kg/cm²; 902 kg/cm².



PROBLEM 9.20

9.21. A steel ring of circular cross-section 20 mm in diameter, having a mean diameter of 50 cm, rotates about the diametral axis OO with a speed of 600 rev/min (see figure). Determine the maximum normal stress in the ring.

Answer: 510 kg/cm².



PROBLEM 9.21

9.22. A solid steel disc of constant thickness, having a diameter of 80 cm, rotates with a constant speed of 3000 rev/min. Determine the maximum normal stress.

Answer: 517 kg/cm².

37. OSCILLATIONS, RESONANCE *

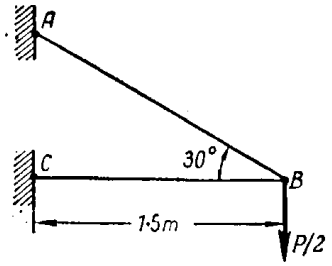
9.23. Determine the period t_0 and the frequency ω_0 of the free longitudinal vibrations of a circular steel column 5 cm in diameter and 80 cm long, loaded by a longitudinal compressive load of 10 t. Solve the problem, first without taking the mass of the column into account, and then with the mass taken into account.

Answer: $t_0 = 0{\cdot}02763$ sec; $\omega_0 = 219{\cdot}4$ sec^{-1}; $t_0 = 0{\cdot}02864$ sec; $\omega_0 = 219{\cdot}3$ sec^{-1}.

9.24. An electric motor with a weight (P) of 1000 kg is mounted on two brackets ABC, each of which is made of wooden bars AB and BC having a 10×10 cm² square cross-section. The bars are hinged at points A, B and C (see figure). Determine the period and the frequency of the free vertical vibrations. The modulus E is 10^5 kg/cm².

* Unless instructed to the contrary, the problems of this paragraph are to be solved without taking into account the mass of elements of construction.

Answer: $t_0 = 0{\cdot}048$ sec; $\omega_0 = 131$ sec^{-1}.



PROBLEM 9.24

9.25. An electric motor fastened to the brackets (see preceding problem) makes 1300 rev/min. As a result of an imperfect balance of the parts of the motor rotating in a plane perpendicular to the axis of rod B, a centrifugal force of 160 kg is induced. Considering only the forced vertical vibrations of the system, calculate the coefficient of dynamic magnification, the impact factor, and the value of the maximum normal stress in the members of the bracket. Neglect the friction forces.

Solution. From the solution of the preceding problem the natural frequency ω_0 of the system is 131 sec^{-1}. The frequency of the exciting force is

$$\omega = \frac{\pi n}{30} = \frac{3{\cdot}14 \times 1300}{30} = 136{\cdot}2 \text{ sec}^{-1}.$$

The coefficient of dynamic magnification

$$\beta = \frac{1}{\left(\frac{\omega}{\omega_0}\right)^2 - 1} = \frac{1}{\left(\frac{136{\cdot}2}{131}\right)^2 - 1} = 12{\cdot}35.$$

The impact factor is calculated from the formula

$$k_d = 1 + \frac{\delta_H}{\delta_Q}\beta,$$

where δ_Q is the static deflection of the bracket due to one half of the weight of the motor, and δ_H is the static deflection of the

bracket due to one half of the maximum value of the exciting force. It is obvious that

$$\frac{\delta_H}{\delta_Q} = \frac{H}{Q} = \frac{80}{500} = 0{\cdot}16.$$

Thus

$$k_d = 1 + (0{\cdot}16 \times 12{\cdot}35) = 2{\cdot}98.$$

The maximum normal stress occurs in bar AB; it is

$$\sigma_{max} = \frac{P_{AB}}{F} k_d = \frac{P}{F} k_d = \frac{1000 \times 2{\cdot}98}{10 \times 10} = 29{\cdot}8 \text{ kg/cm}^2.$$

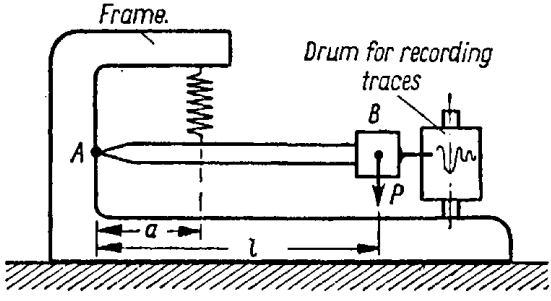
9.26. A helical steel spring having 20 coils with a mean coil diameter of 12 cm, and a wire diameter of 1 cm, is loaded by a weight of 20 kg. The spring weighs 4·8 kg. The modulus of rigidity G of the material of the spring is 8×10^5 kg/cm^2. Determine the period of free longitudinal oscillations of the load supported by the spring (i) without the mass of the spring taken into account and (ii) with the mass of the spring taken into account.

Answer: (i) 0·53 sec; (ii) 0·55 sec.

9.27. Two identical weights are suspended from a helical spring by means of a hook. Both weights together will extend the spring by 2·5 cm. One of the weights is taken off suddenly. Determine the period of the longitudinal oscillations set up, and the maximum value of the velocity and the acceleration of the oscillating load.

Answer: 0·224 sec; 35 cm/scc; 981 cm/sec^2.

9.28. Determine the value of the load P for the vibrograph represented diagrammatically in the figure. It is required that the



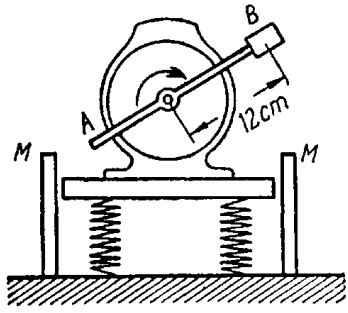
PROBLEM 9.28

period of oscillation of this load, with the spring having a stiffness of 0·25 kg/cm, should be 2 sec. The ratio a/l of the lever arms is 0·4. In the calculation neglect the weight of lever AB, and also its deformation.

Answer: 9·95 kg.

9.29. To demonstrate the phenomenon of resonance an electric motor weighing 8 kg is mounted on four helical springs, each having nine coils with a mean coil diameter of 5 cm, and a wire diameter of 6 mm. The modulus, G, is 8×10^5 kg/cm^2. A rod AB, carrying at end B a weight, Q, of 150 g located at a distance of 12 cm from the axis of the electric motor, is fixed on to the shaft of the electric motor (see figure). The horizontal displacement of the frame of the electric motor is prevented by guides M.

Determine at what speed of the electric motor resonance will take place. What will then be the amplitude of vibrations, and what will be the maximum shear stress in the springs? In the calculation take into account the friction force, assuming it to be proportional to the velocity of the vibrational motion. The damping factor n, of the oscillations is to be taken as 7 sec^{-1}.



PROBLEM 9.29

Solution. The static deflection of each spring, due to one quarter of the weight of the electric motor is

$$\lambda_s = \frac{4 \times \frac{P}{4} R^3 n}{Gr^4} = \frac{4 \times 2 \times 2{\cdot}5^3 \times 9}{8 \times 10^5 \times 0{\cdot}3^4} = 0{\cdot}174 \text{ cm}.$$

The natural frequency of the electric motor on the springs

$$\omega_0 = \sqrt{\frac{g}{\lambda_s}} = \sqrt{\frac{981}{0{\cdot}174}} = 75{\cdot}2 \text{ sec}^{-1}.$$

Centrifugal force is set up by the rotation of the load $Q = 150$ g. Resonance will take place when the frequency, ω, of the exciting force coincides with the natural frequency ω_0. Therefore the critical speed is

$$n_{\text{crit}} = \frac{30\omega}{\pi} = \frac{30\,\omega_0}{\pi} = \frac{30 \times 75{\cdot}2}{3{\cdot}14} = 718 \text{ rev/min}.$$

The maximum value of the exciting force H, equal to the centrifugal force Q_d, is

$$H = Q_d = \frac{Q}{g}\omega^2 r = \frac{Q}{g}\omega_0^2 r = \frac{0{\cdot}15}{981} \times 75{\cdot}2^2 \times 12 = 10{\cdot}37 \text{ kg}.$$

The coefficient of dynamic magnification in the given case ($\omega = \omega_0$) is determined from the formula

$$\beta = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4\left(\frac{n}{\omega_0}\right)^2\left(\frac{\omega}{\omega_0}\right)^2}} = \frac{\omega_0}{2n} = \frac{75{\cdot}2}{2 \times 7} = 5.37.$$

The impact factor is

$$k_d = 1 + \frac{\delta_H}{\delta_P}\beta = 1 + \frac{H}{P}\beta = 1 + \frac{10{\cdot}37}{8} \times 5{\cdot}37 = 7{\cdot}96.$$

The amplitude of the forced vibrations is

$$A = \delta_H\beta = \frac{H}{P}\beta\,\delta_P = \frac{H}{P}\beta\,\lambda_s = \frac{10{\cdot}37}{8} \times 5{\cdot}37 \times 0{\cdot}174 = 1{\cdot}21 \text{ cm}.$$

The maximum shear stress in the springs is

$$\max \tau_d = \max \tau_s k_d = \frac{2 \times \frac{P}{4} R}{\pi r^3}\left(1 + \frac{r}{2R}\right) k_d$$

$$= \frac{2 \times 2 \times 2.5}{3{\cdot}14 \times 0{\cdot}3^2}\left(1 + \frac{0{\cdot}3}{5}\right) \times 7{\cdot}96 = 995 \text{ kg/cm}^2.$$

9.30. Find the period of free torsional oscillations of a steel shaft 12 cm in diameter and 150 cm long, one end of which is clamped, and the other end of which carries a pulley with a moment of inertia J_0 of 8000 kg cm sec^2.

Answer: 0·171 sec.

9.31. A flywheel, weighing 2·5 t and having a radius of gyration of 110 cm, is mounted on the end of a steel shaft 110 mm in diameter and 2 m long. The other end of the shaft is clamped.

Determine the period of free torsional vibrations. At what rotational speed of the flywheel can the torsional vibrations be dangerous, if the other end of the shaft is connected to a crank subjected to two impulses per each revolution?

Answer: 0·46 sec; 65·2 rev/min.

9.32. A simply supported No. 27a I-beam, with a length of 6 m, carries a load of 3 t at its mid-span.

Determine the natural frequency of the system (i) without the mass of the beam taken into account, and (ii) with the mass of the beam taken into account.

Answer: 30·9 sec^{-1}; 30·2 sec^{-1}.

9.33. The period of free vibrations (determined experimentally) of a load, weighing 20 kg, when fastened to the end of a wooden cantilever 1 m in length, was found to be 0·2 sec. The beam has a rectangular cross-section 6 cm in depth and 4 cm in width. The depth of the beam is parallel to the plane of action of the load.

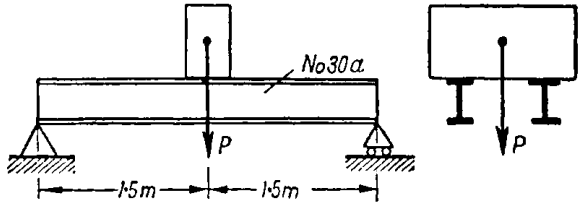
Determine the modulus of elasticity of the material of the beam. Neglect the weight of the beam.

Answer: 0·93 $\times$ 10^5 kg/cm^2.

9.34. A motor with a weight of 1·2 t is fixed at the middle of the span, l (= 3 m), of two No. 30a I-beams, both of which are simply supported (see figure).

Determine the dynamic magnifier, the impact factor, and the value of the maximum normal stress in the beam, if the motor rotates at 2000 rev/min, and the centrifugal force H, set up by an imperfect balance of the rotating parts of the motor, is 200 kg.

Take the weight of the beam into account, and neglect friction forces.



PROBLEM 9.34

Solution. The natural frequency of the load is determined by the formula

$$\omega_0 = \sqrt{\frac{g}{f_s}} = \sqrt{\frac{48 E J_a g}{\left(\frac{1}{2} P + \frac{17}{35} Q\right) l^3}} = 215{\cdot}9 \text{ sec}^{-1}.$$

The frequency of the exciting force is

$$\omega = \frac{\pi n}{30} = \frac{3{\cdot}14 \times 2000}{30} = 209{\cdot}4 \text{ sec}^{-1}.$$

The dynamic magnifier

$$\beta = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2} = \frac{1}{1 - \left(\frac{209{\cdot}4}{215{\cdot}9}\right)^2} = 17{\cdot}04.$$

The impact factor

$$k_d = 1 + \frac{f_H}{f_P} \beta = 1 + \frac{H}{P} \beta = 1 + \frac{200}{1200} \times 17{\cdot}04 = 3{\cdot}84.$$

The maximum normal stress in the beams

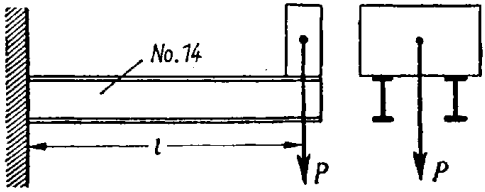
$$\max \sigma_d = \max \sigma_s k_d = \frac{Pl}{2 \times 4 \times W} k_d = \frac{1200 \times 300}{2 \times 4 \times 597} \times 3{\cdot}84$$

$$= 289 \text{ kg/cm}^2.$$

9.35. An electric motor 240 kg in weight, fixed to the ends of two cantilevers of I-section No. 14 (see figure), makes 600 rev/min. During running of the motor a centrifugal force of 40 kg is set up.

Determine the maximum normal stress in the beams, if their length is 2 m. Determine the length of the beams at which resonance takes place, and the corresponding maximum normal stress in the beams. In the calculation take into account the force of resistance, which may be assumed proportional to the velocity of the oscillatory motion; assume that the coefficient (n) of damping is 2/sec. Neglect the mass of the beam. The modulus, E, is 2×10^6 kg/cm^2.

Answer: 583 kg/cm^2; 207 cm; 881 kg/cm^2.



PROBLEM 9.35

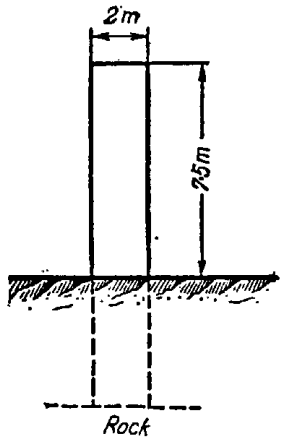
9.36. A motor 180 kg in weight, rotating at 1200 rev/min, is fixed on to a simply supported beam at the middle of the span l (= 3 m). Choose the I-section of the beam so that its natural frequency would be approximately 20% greater than the frequency of the exciting force. Assuming that the maximum value of the exciting force is 100 kg, find the maximum normal stress in the beam. Neglect the mass of the beam, and the force of friction.

Answer: I-section No. 16; $\sigma_{max} = 286$ kg/cm^2.

9.37. A stonework pillar of a 2×2 m^2 square transverse section rests upon rock located at an unknown depth below the surface of the ground (see figure). The modulus of elasticity of the material is not known. The specific weight of the masonry of the pillar is approximately 2 t/m^3. It was found possible to determine experimentally the deflection of the upper end of the pillar due to the action of a horizontal force P of 10 t; this was 7 mm. Further, it was found possible to determine the period of the free transverse vibrations of the column (without the force P); this was 0·25 sec.

Assuming that the pillar is set into the rock, and that the earth around it exerts no influence on the behaviour of the pillar, determine the depth at which the rock is located, and the modulus of elasticity of the stonework.

Answer: 4·27 m; 58,200 kg/cm².



PROBLEM 9.37

38. STRESSES AND DEFORMATIONS ON IMPACT*

9.38. A steel rod, 25 mm in diameter and 1·5 m in length, is subjected to a suddenly applied tensile force of 2·5 t. Determine the maximum stress, and the maximum deformation.

Answer: 1020 kg/cm²; 0·765 mm.

9.39. A vertical rod, statically compressed by a force P, is shortened by 2 mm. Determine the maximum reduction in the length of the rod, if the same load falls on to it from a height of 1 mm.

Answer: 4·83 mm.

* Unless instructed to the contrary, problems in this section are to be solved without the mass of elements of construction taken into account. In cases of impact due to a falling load the calculation of impact factors is to be effected by accurate formulae. (See the book *Strength of Materials* by N. M. Belyayev, 1954 Edition; the formulae (36.10), (36.11), and (36.12) of § 226.)

9.40. A round steel bar, 5 m long, is suspended from its upper end, and provided with a shoulder at its lower end. A load of 800 kg falls along the bar from a height of 16 mm, and is stopped by the shoulder.

Determine the required diameter of the bar, if the allowable stress is 1600 kg/cm². To calculate the stress on impact apply the accurate formula, and also the following approximate formula

$$\sigma_d = \sigma_s \sqrt{\frac{2H}{\Delta l_s}}.$$

Answer: 2·53 cm; 2·26 cm.

9.41. A load of 1·5 t falls along a vertical steel rod which is suspended by its upper end and provided with a shoulder at its lower end. The length of the rod is 4 m, its diameter is 4 cm. To soften the impact the shoulder is placed on a helical spring, which deflects 0·625 mm under a static load of 100 kg.

Determine the height, measured from the top of the unloaded spring, from which the load must fall in order to produce a stress of 1200 kg/cm² in the rod. Determine the height from which the load must fall in the absence of the spring.

Answer: 38·9 cm; 0·97 cm.

9.42. A load P, of 5 t is fastened to a steel cable the other end of which is wound on the drum of a winch. The load is lowered with a constant velocity, v, of 1·6 m/sec. On a sudden braking of the winch the load is stopped instantly. At the instant of stopping, the length, l, of the cable between the load and the winch is 240 m. The cross-sectional area, F, of the cable is 10 cm². Determine the maximum stress in the cable (a) without taking its weight into account, and (b) with its weight taken into account.

Solution. To calculate the maximum stress in the cable due to a sudden braking of the winch (neglecting the weight of the cable), we make use of the formula

$$\sigma_d = \sigma_s k_d = \frac{P}{F}\left[1 + \sqrt{\frac{v^2 E F}{g P l}}\right].$$

Inserting the numerical data, we obtain

$$\sigma_d = \frac{5000}{10}\left[1 + \sqrt{\frac{160^2 \times 2 \times 10^6 \times 10}{981 \times 5000 \times 24{,}000}}\right] = 1543 \text{ kg/cm}^2.$$

On taking the weight of the cable into account the impact factor must be calculated by the formula

$$k'_d = 1 + \sqrt{\frac{v^2 EF}{gPl(1+\beta)}},$$

where β is the ratio of the effective mass of the cable, $Q_c/3g = \gamma Fl/3g$, to the mass of the load, P/g, i.e.

$$\beta = \frac{\gamma F l g}{3gP} = \frac{\gamma Fl}{3P} = \frac{0{\cdot}0078 \times 10 \times 24{,}000}{3 \times 5000} = 0{\cdot}125.$$

The maximum stress in the cable is the sum of the static stress due to the weight of cable, $\sigma'_s = \gamma Fl/F = \gamma l$, and the dynamic stress due to the braking of the winch, $\sigma_d = \sigma_s k'_d$. Thus

$$\sigma = \sigma'_s + \sigma_s k'_d = \gamma l + \frac{P}{F}\left[1 + \sqrt{\frac{v^2 EF}{g\,Pl(1+\beta)}}\right]$$

$$= (0{\cdot}0078 \times 2400) + \frac{5000}{10}$$

$$\times \left[1 + \sqrt{\frac{160^2 \times 2 \times 10^6 \times 10}{981 \times 5000 \times 24{,}000(1 + 0{\cdot}125)}}\right]$$

$$= 187 + 1483 = 1670 \text{ kg/cm}^2.$$

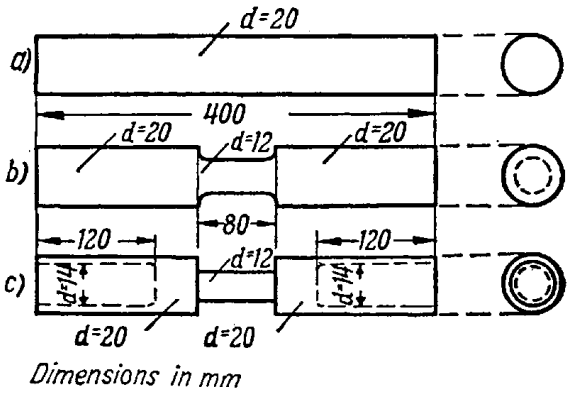
9.43. At the end of the cable of a winch, having a cross-sectional area of 6 cm^2, is suspended a load of 800 kg, which is being lowered with a constant velocity of 36 m/min. A helical spring, having 12 coils with a mean coil diameter of 12 cm and a rod diameter of 2·4 cm, is placed between the load and the cable. When the length of the unwound cable is 9 m a sudden braking of the winch occurs. Determine the maximum stress in the cable, the maximum shear stress in the spring, and the elongation of the spring at the moment of braking (see the solution of the preceding problem). Neglect the weight of the cable, and that of the spring. The modulus, G, for

the material of the spring is 8×10^5 kg/cm^2; the modulus, E, for the cable is 2×10^6 kg/cm^2.

Answer: 247 kg/cm^2; 3602 kg/cm^2; 9·26 cm.

9.44. Compare the values of the maximum tensile stresses in three steel rods (see figure) subjected to a longitudinal impact by a weight of 250 kg. At the moment of impact the weight possesses 20 kg cm of kinetic energy.

Answer: $\sigma_a = 881$ kg/cm^2; $\sigma_a : \sigma_b : \sigma_c = 1 : 2{\cdot}43 : 2{\cdot}08$.



PROBLEM 9.44

9.45. A helical spring working in compression is made of steel wire 6 mm in diameter. The mean diameter of the coil is 12 cm; the number of coils is 18.

Determine the value of a static load which will compress the spring 2·5 cm. Determine the deflection of the spring and the maximum shear stress in it, assuming that the same load will fall on to the spring (unloaded) from a height of 10 cm.

Answer: 1·04 kg; 10 cm; 604 kg/cm^2.

9.46. Determine the required number of coils of a helical steel spring, having a mean coil diameter of 15 cm and a rod diameter of 3 cm. The spring must be sufficiently strong (with respect to the limiting stress) to arrest a body of 200 kg, moving horizontally with a velocity of 2 m/sec. The allowable shear stress $[\tau]$ is 2100 kg/cm^2.

Answer: 10·8.

9.47. A solid disc, 1 m in diameter and 10 mm thick, rotating with a constant angular velocity $\omega = 5\ \text{sec}^{-1}$, is mounted on a shaft 6 cm in diameter.

Determine the value of the maximum shear stress in the cross-section of the shaft, if rotation of the shaft is stopped instantly at a section located at a distance of 2 m from the disc.

Solution. Neglecting the mass of the shaft, the maximum shear stress in it can be calculated by the formula

$$\tau_{\max} = \omega \sqrt{\frac{2 J_0 G}{F l}} = \frac{\omega}{d} \sqrt{\frac{8 J_0 G}{\pi l}},$$

where $J_0 = \pi D^4 t\gamma/32g$ is the moment of inertia of the disc (D is the diameter, t is the thickness of the disc). Inserting the numerical data we find:

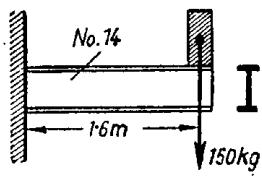
$$\tau_{\max} = \frac{\omega D^2}{2d} \sqrt{\frac{G t \gamma}{g l}} = \frac{5 \times 10^4}{2 \times 6} \sqrt{\frac{8 \times 10^5 \times 1 \times 0{\cdot}0078}{981 \times 200}}$$
$$= 745\ \text{kg/cm}^2.$$

9.48. Two flywheels, with equal moments of inertia of 3000 kg cm sec^2, are mounted on a steel shaft 120 mm in diameter. The distance between the flywheels is 4 m. One flywheel is permantly fixed to the shaft, whereas the other, rotating freely at 40 rev/min, can be engaged instantaneously (by means of a special device) with the shaft, and thus bring the whole system into motion.

Determine the maximum shear stress in the cross-section of the shaft at the instant the other flywheel engages with it.

Answer: 965 kg/cm^2.

9.49. A load weighing 150 kg is applied suddenly, without an initial velocity, to the free end of an I-section cantilever 1·6 m long



PROBLEM 9.49

(see figure). Determine the deflection of the beam at the point of application of the load, and the maximum normal stress at the fixed end.

Answer: 0·288 cm; 471 kg/cm².

9.50. A load of 4·5 t, applied to the mid-span of a simply supported beam, causes a deflection of 2 cm. With what maximum velocity may a load of 500 kg fall on to this beam, without causing a deflection greater than 2 cm?

Solution. The static deflection of the beam, caused by the 500 kg load, will obviously be

$$f_s = 2 \times \frac{500}{4500} = 0{\cdot}222 \text{ cm}.$$

Since $f_d = K_d f_s = 2$ cm, then $K_d = f_d/f_s = 2/0{\cdot}222 = 9$; also

$$K_d = 1 + \sqrt{1 + \frac{2H}{f_s}} = 1 + \sqrt{1 + \frac{v^2}{g f_s}}$$

$$= 1 + \sqrt{1 + \frac{v^2}{981 \times 0{\cdot}222}} = 9.$$

Hence

$$v = \sqrt{[(9-1)^2 - 1] \times 981 \times 0{\cdot}222} = 117 \text{ cm/sec}.$$

9.51. A model of a simply supported steel beam, made of the same steel, and one-fifth of the full size, was tested in impact by a concentrated load 5 times smaller than the dynamic load on the actual beam. The height of fall of this load was also 5 times less than the actual one. The impact factor which was determined experimentally, by comparing the static and dynamic deformations of the model, was found to be 6. Determine the value of the impact factor for the actual beam.

Answer: 12.

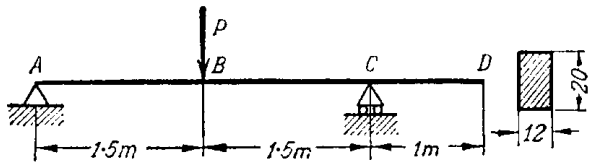
9.52. A load of 45 kg falls from a height of 10 cm on to a beam which deflects 2·5 cm under this impact. What static load applied to the same section of the beam will cause the same deflection?

Answer: 450 kg.

9.53. A wooden beam of a 12×20 cm^2 cross-section (see figure) was subjected to flexural impact by a load P of 100 kg at section B. The maximum deflection of the beam at section D, on impact, was found to be 6 mm.

Determine the height of fall of the load, and the maximum normal stress in the beam. The modulus, E, is 10^5 kg/cm^2.

Answer: 1·96 cm; 80 kg/cm^2.



PROBLEM 9.53

9.54. As a result of an impact load on the end of a cantilever 180 cm long, the maximum normal stress in the beam was found to be 3·5 times greater than that due to the static action of the same load. By how much will the deflection of the free end of the beam be reduced, if the point of impact is transferred 20 cm nearer to the fixed end?

Answer: By 6·8%.

9.55. Compare the maximum normal stresses and the maximum deflections of a steel beam, having a 2×3 cm^2 cross-section (the depth is parallel to the direction of the load), with those of a steel leaf spring, consisting of leaves 50 mm wide and 6 mm thick. The spring has, in the critical section, a flexural resisting moment equal to that of the beam. The beam and the leaf spring, both 1 m in span, are simply supported, and subjected to an impact at the mid-span by a load of 15 kg falling from a height of 1 cm.

Answer: Beam: 1082 kg/cm^2 and 0·3 cm; leaf spring: 493 kg/cm^2 and 1·03 cm.

9.56. A No. 40a I-beam, with a 6 m span, is simply supported. The web of the I-section is vertical, and a load of 400 kg falls from a height of 5 cm on to the middle of the span.

Determine the maximum normal stress and the maximum deflection of the beam. Solve the problem without taking the mass

of the beam into account, and also with the mass of the beam taken into account.

Answer: 873 kg/cm² and 0·653 cm; 734 kg/cm² and 0·555 cm.

9.57. Determine by how many times the maximum stress in a No. 60a I-section cantilever will increase, if the web of the I-section is placed horizontally (instead of being placed vertically). The beam has a length of 4 m and is loaded at its free end by a vertical concentrated load of 200 kg.

Compare these results (calculated for the static load) with those calculated for the load falling on to the beam from a height of 5 cm. In the calculations take the weight of the beam into account.

Answer: With the static load: the maximum stresses increase 904/62·3 = 14·5 times; with impact: the maximum stresses increase 1929/518 = 3·72 times.

9.58. Determine the diameter of a simply supported wooden beam of circular cross-section, having a length of 3 m. A load of 150 kg, having a velocity of 50 cm/sec at the instant of impact, falls on to the beam at mid-span. The allowable stress is 80 kg/cm²; the maximum allowable deflection is $l/400$; the modulus, E, is 8×10^4 kg/cm². Solve the problem (i) using the accurate formula to calculate the impact factor, and (ii) using the following approximate formula to calculate the impact factor:

$$K_d = \sqrt{\frac{2H}{f_s}}.$$

Solution. Using the approximate formula for the calculation of the impact factor we can write

$$\max \sigma_d = \max \sigma_s K_d = \frac{Pl}{4W}\sqrt{\frac{2H}{f_s}} = \frac{Pl}{4W}\sqrt{\frac{48EJv^2}{gPl^3}}$$

$$= \frac{1}{d}\sqrt{\frac{48PEv^2}{g\pi l}} \leq [\sigma]$$

and

$$\max f_d = \max f_s K_d = \frac{Pl^3}{48EJ}\sqrt{\frac{48EJv^2}{gPl^3}} = \frac{1}{d^2}\sqrt{\frac{4Pl^3v^2}{3gE\pi}} \leq [f],$$

whence

$$d \geq \frac{1}{[\sigma]}\sqrt{\frac{48\,P\,E\,v^2}{g\pi l}} = \frac{1}{80}\sqrt{\frac{48 \times 160 \times 8 \times 10^4 \times 50^2}{981 \times 3{\cdot}14 \times 300}} = 25{\cdot}8 \text{ cm}$$

and

$$d \geq \sqrt{\frac{1}{[f]}}\sqrt{\frac{4Pl^3v^2}{3gE\pi}} = \sqrt{\frac{400}{300}}\sqrt{\frac{4 \times 160 \times 300^3 \times 50^2}{3 \times 981 \times 8 \times 10^4 \times 3{\cdot}14}}$$

$$= 18 \text{ cm}.$$

It will be necessary to take $d = 25{\cdot}8 \approx 26$ cm.

Using the accurate formula for the calculation of the impact factor we have

$$\max \sigma_d = \max \sigma_s K_d = \frac{Pl}{4W}\left[1 + \sqrt{1 + \frac{48\,E\,J\,v^2}{gPl^3}}\right]$$

$$= \frac{122{,}230}{d^3}\left[1 + \sqrt{1 + 0{\cdot}0001112\,d^4}\right] \leq [\sigma] = 80 \text{ kg/cm}^2.$$

Thus for the determination of d we have the equation

$$d^3 - 259{\cdot}6d - 3056 = 0,$$

whence $d = 20{\cdot}3$ cm. (The approximate formula thus gives a considerably greater diameter for the beam.) For the accurate formula

$$J = \frac{\pi d^4}{64} = \frac{3{\cdot}14 \times 20{\cdot}3^4}{64} = 8270 \text{ cm}^4$$

and

$$\max f_d = \frac{Pl^3}{48EJ}\left[1 + \sqrt{1 + \frac{48\,E\,J\,v^2}{gPl^3}}\right]$$

$$= \frac{160 \times 300^3}{48 \times 8 \times 10^4 \times 8270}\left[1 + \sqrt{1 + (0{\cdot}000112 \times 20{\cdot}3^4)}\right]$$

$$= 0{\cdot}74 \text{ cm} < [f] = \frac{l}{400} = \frac{300}{400} = 0{\cdot}75 \text{ cm}.$$

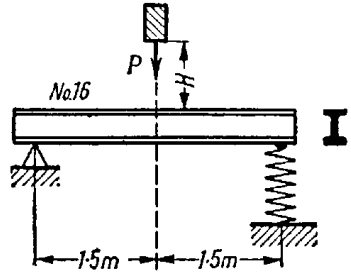
9.59. A No. 18 I-beam is simply supported and has a span of 2 m. Determine the maximum normal stress and the maximum deflection of the beam when a load of 600 kg, having a velocity of 50 cm/sec at the instant of impact, falls on to the beam at mid-span. Solve the problem assuming that (a) the supports of the beam are absolutely rigid, and (b) rubber pads 10 cm wide, 15 cm long and 15 mm thick are inserted between the supports and the beam. The modulus E for rubber is 80 kg/cm^2.

Answer: (a) 1665 kg/cm^2, 0·309 cm; (b) 1170 kg/cm^2, 0·217 cm.

9.60. One end of a No. 16 I-beam, 3 m long, rests on a rigid pin-jointed support, and the other end on a helical steel spring having 10 coils, with a mean coil diameter of 10 cm, and a wire diameter of 20 mm (see figure).

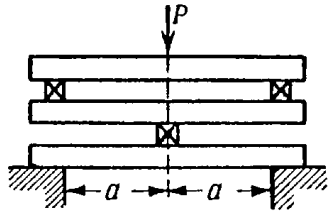
If $[\sigma] = 1600$ kg/cm^2 (for the material of the beam) and $[\tau] = 2000$ kg/cm^2 (for the material of the spring), from what height (H) may the load P (= 200 kg) fall on to the beam, without stresses in the beam and spring exceeding the allowable values?

Answer: 3·84 cm.



PROBLEM 9.60

9.61. Three identical steel beams are arranged as indicated in the figure. Each beam has a 6 × 6 cm^2 cross-section and a length of



PROBLEM 9.61

120 cm. Determine (a) the weight of a body falling from a height of 1 cm to cause a maximum normal stress of 800 kg/cm² in the beams, and (b) the normal stress which would be caused by the load falling on to the single bottom beam. The pads between the beams are assumed to be non-deformable.

Answer: (a) 155·7 kg; (b) 1275 kg/cm².

9.62. A load of 10 kg falls from a height of 6 cm on to the free end of a cantilever having a length of 1·5 m. The beam has a circular cross-section, with a diameter of 6 cm over one half of its length, and a diameter of 4 cm over the other half.

Determine the maximum normal stress and the maximum deflection of the beam on impact, assuming that: (a) the free end of the beam has the smaller diameter; (b) the free end of the beam has the larger diameter.

Answer: (a) 1258 kg/cm²; 1·41 cm; (b) 1564 kg/cm²; 2·64 cm.

9.63. A steel rod, 5 mm in diameter and 1 m long, remains horizontal while falling from a height of 10 cm, and hits rigid supports simultaneously at both ends.

Determine the maximum normal stress in the material of the rod, assuming that the kinetic energy at the moment of impact is equal to the potential energy of deformation of the rod due to the impact.

Answer: 1530 kg/cm².

9.64. Two concentric helical springs, working together, are subjected to a longitudinal impact by a load of 5 kg falling from a height of 10 cm. Determine the deflection of the springs and the maximum shear stresses in them, if the characteristics of the springs are: (A) the mean coil diameter is 6 cm, the wire diameter is 4 cm, the number of coils is 12; (B) the mean diameter of coil is 8 cm, the wire diameter is 8 mm, the number of coils is 6. The modulus, G, is 8×10^5 kg/cm² for both springs.

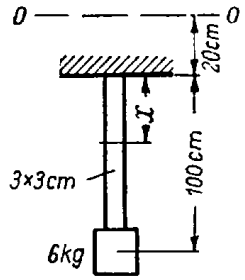
Answer: 3·01 cm; $\tau_A = 734$ kg/cm²; $\tau_B = 1679$ kg/cm².

9.65. A 1 m long steel bar of a 3×3 cm² cross-section, clamped at one end and carrying at the other end a load of 6 kg, rotates

together with the clamping arrangement about the horizontal OO axis with a speed of 10 rev/min (see figure).

Determine the maximum normal stress in the bar, and its maximum deflection on a sudden stoppage of the clamping arrangement. Neglect the weight of the bar.

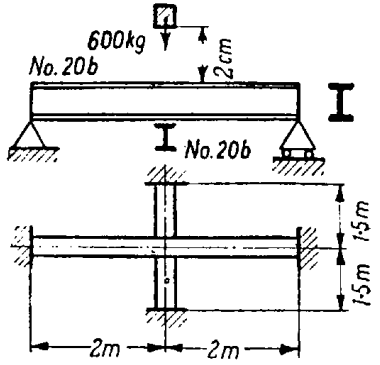
Answer: 1390 kg/cm²; 1·54 cm.



PROBLEM 9.65

9.66. A load weighing 600 kg falls from a height of 2 cm on to two contacting simply supported No. 20 b I-beams lying cross-wise (see figure). Determine the maximum normal stresses in the beams on impact.

Answer: In the lower beam: 1296 kg/cm², in the upper beam: 728 kg/cm².

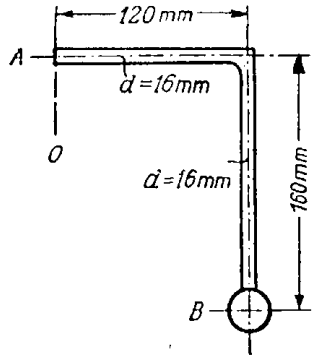


PROBLEM 9.66

9.67. A steel bent, encastré-ended at section A and carrying at end B a load weighing 3 kg, rotates with a constant speed of 40 rev/min about the OO axis (see figure).

Determine the maximum effective stress in the rod if rotation is stopped suddenly. Apply the third theory of strength.

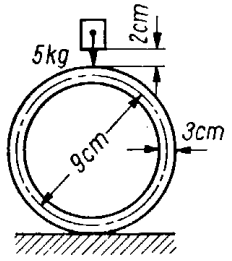
Answer: 1461 kg/cm².



PROBLEM 9.67

9.68. Determine the maximum normal stress in a steel ring of rectangular cross-section, 3 cm high and 4 cm wide, if a load of 5 kg falls from a height of 2 cm (see figure) in the diametral direction. In the solution of the problem take into account only the energy of flexure.

Answer: 1458 kg/cm².



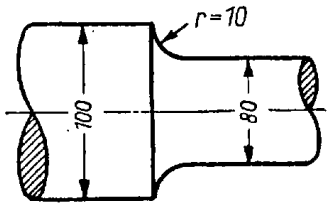
PROBLEM 9.68

39. REPEATED STRESSES

9.69. A rotating stepped shaft is subjected to a constant bending moment M. The shaft is made of a carbon steel with an ultimate strength $\sigma_u = 45$ kg/mm², and a limit of endurance in flexure $\sigma^b_{-1} = 22$ kg/mm² (symmetrical cycle). The diameters of the shaft

are: $D = 100$ mm and $d = 80$ mm; the fillet has a radius $r = 10$ mm (see figure).

Determine the maximum allowable value of bending moment M. Take the factor of safety k, with respect to the endurance limit, as 2.



PROBLEM 9.69

Solution. The value of the limit of endurance of a component may be determined by the formula

$$\sigma_{-1}^{bc} = \frac{\sigma_{-1}^{b}}{\alpha_{af}\alpha_s},$$

where σ_{-1}^{b} is the limit of endurance determined experimentally on a small scale model, α_{af} is the actual stress concentration factor for the small scale model, and α_s is the scale factor. The value of α_{af} in its turn may be determined by the formula

$$\alpha_{af} = 1 + q(\alpha_{tf} - 1),$$

where α_{tf} is the theoretical stress concentration factor, and q is the notch sensitivity of the material, depending fundamentally on the value of α_{tf} and the ultimate strength of the material.

The value of α_{tf} may be determined from Table 37* in the book *Strength of Materials* by N. M. Belyayev, 1954 Edition. At $r/d = 10/80 = 0{\cdot}125$ we find that $\alpha_{tf} = 1{\cdot}5$. From Fig. 626* (see the above-mentioned book) we find (by means of a linear interpolation) that at this value of α_{tf} the notch sensitivity q, for a steel having σ_u equal to 45 kg/mm^2, is 0·36.

Thus

$$\alpha_{af} = 1 + 0{\cdot}36(1{\cdot}5 - 1) = 1{\cdot}18.$$

* See Appendix B, Section 6.

The value of the coefficient α_s for a component of carbon steel, with a moderate stress concentration, is determined from curve 2 of Fig. 629 (see Appendix B, Section 6). At $d = 80$ mm we have $\alpha_s = 1{\cdot}56$. The limit of endurance of the component (the shaft) in a symmetrically varying stress cycle is

$$\sigma_{-1}^{bc} = \frac{2200}{1{\cdot}18 \times 1{\cdot}56} = 1195 \text{ kg/cm}^2.$$

The allowable stress

$$[\sigma_{-1}^{bc}] = \frac{\sigma_{-1}^{bc}}{k} = \frac{1195}{2} = 597 \text{ kg/cm}^2.$$

The maximum allowable value of the bending moment M is determined from the condition of limiting stress

$$\max \sigma = \frac{M_{max}}{W} = \frac{32 M_{max}}{\pi d^3} \leq [\sigma_{-1}^{bc}],$$

whence

$$M_{max} = \frac{[\sigma_{-1}^{bc}] \pi d^3}{32} = \frac{597 \times 3{\cdot}14 \times 8^3}{32} = 30{,}000 \text{ kg cm} = 300 \text{ kg m}.$$

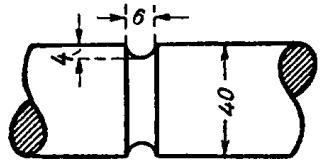
9.70. Solve the preceding problem with the assumption that the fillet radius is 20 mm. The value of the scale factor is to be determined from the curve 1 of Fig. 629 (see Appendix B, Section 6), since the actual stress concentration differs very little from unity.

Answer: 393 kg m.

9.71. Determine the maximum allowable value P_{max} of a longitudinal force for a round bar 40 mm in diameter, with a semicircular recess 4 mm deep (see figure) and 3 mm in radius. The load on the bar varies from $-P_{max}$ (compression) to $+P_{max}$ (tension). The bar is made of an alloy steel having an ultimate strength σ_u of 100 kg/mm^2, and limit of endurance (in tension-compression) σ_{-1}^0 of 32 kg/mm^2 (symmetrical cycle). Assume that the factor of safety with respect to the endurance limit is 1·8. Make use of the

data of Table 37 and the graphs of Figs. 626 and 629 (see Appendix B, Section 6).

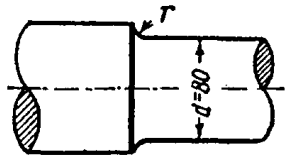
Answer: 4970 kg.



PROBLEM 9.71

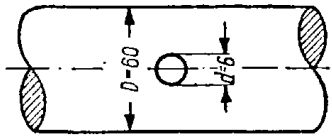
9.72. Determine the radius r of the fillet for a round stepped shaft 80 mm in diameter (see figure), which is twisted by a torque varying from $-M_{max} = 0{\cdot}55$ tm to $+M_{max} = 0{\cdot}55$ tm. The shaft is made of an alloy steel having an ultimate strength σ_u of 100 kg/mm² and limit of endurance in torsion τ^t_{-1} of 25 kg/mm² (symmetrical cycle). Take the factor of safety with respect to the endurance limit equal to two. Make use of the data in Table 37 and the graphs of Figs. 626 and 629 (see Appendix B, Section 6).

Answer: 8 mm.



PROBLEM 9.72

9.73. A round bar 60 mm in diameter, weakened by a round transverse hole 6 mm in diameter (see figure), is subjected to a compressive load varying from zero to a maximum value P_{max}. The bar is made of a carbon steel with an ultimate strength $\sigma_u = 60$ kg/mm². The yield point σ^0_y and endurance limit (in tension–compression) σ^0_{-1} are 28 kg/mm² and 20 kg/mm² respectively (symmetrical cycle).



PROBLEM 9.73

Determine the maximum allowable value of P_{max} if the factor of safety is 1·7.

Solution. From Table 37*, with the ratio of the hole diameter to bar diameter equal to $d/D = 6/60 = 0{\cdot}1$, the theoretical stress concentration factor α_{tf} is 2. From the curve of Fig. 626* we determine the value of the notch sensitivity q. At $\sigma_u = 60$ kg/mm² and $\alpha_{tf} = 2$ we find that q is 0·57. The actual stress concentration for the small scale testpiece is calculated by the formula

$$\alpha_{af} = 1 + q(\alpha_{tf} - 1) = 1 + 0{\cdot}57(2 - 1) = 1{\cdot}57.$$

The value of the scale factor is determined from curve 2 of Fig. 629* with $D = 60$ mm we have $\alpha_s = 1{\cdot}46$.

The allowable stress for the component under a symmetrical cycle of stress variation

$$[\sigma_{-1}^{0c}] = \frac{\sigma_{-1}^{0c}}{k} = \frac{\sigma_{-1}^{0}}{\alpha_{af}\alpha_s k} = \frac{2000}{1{\cdot}57 \times 1{\cdot}46 \times 1{\cdot}7} = 513 \text{ kg/cm}^2.$$

The allowable stress under a constant load

$$[\sigma_{+1}^{0c}] = \frac{\sigma_y^0}{k} = \frac{2800}{1{\cdot}7} = 1647 \text{ kg/cm}^2.$$

The value of the allowable stress for a non-symmetrical cycle of stress is determined by the approximate formula, obtained by considering the straight-line diagram of the allowable stresses within the co-ordinates $\sigma_{max} - \sigma_m$, or $\sigma_a - \sigma_m$:

$$[\sigma_r^{0c}] = \frac{2\,[\sigma_{-1}^{0c}]\,[\sigma_{+1}^{0c}]}{(1 + r)[\sigma_{-1}^{0c}] + (1 - r)[\sigma_{+1}^{0c}]}.$$

Since in the given case the characteristic of the cycle $r = \sigma_{min}/\sigma_{max} = P_{min}/P_{max} = 0$, then

$$[\sigma_r^{0c}] = [\sigma_0^{0c}] = \frac{2 \times 513 \times 1647}{(1 + 0)513 + (1 - 0)1647} = 782 \text{ kg/cm}^2.$$

* See Appendix B, Section 6.

The maximum allowable value of the load

$$P_{\max} = [\sigma_0^{0c}]F = [\sigma_0^{0c}]\left(\frac{\pi D^2}{4} - dD\right)$$

$$= 782\left[\frac{3{\cdot}14 \times 6^2}{4} - (0{\cdot}6 \times 6)\right] = 19{,}300 \text{ kg}.$$

9.74. Solve the preceding problem, assuming that the bar is made of an alloy steel with $\sigma_u = 90$ kg/mm^2, $\sigma_y^0 = 75$ kg/mm^2 and $\sigma_{-1}^0 = 29$ kg/mm^2.

Answer: 22,750 kg (an increase of 18%).

9.75. A round bar 85 mm in diameter is made of a carbon steel, for which σ_u is 65 kg/mm^2, σ_y^b is 40 kg/mm^2, and σ_{-1}^b is 28 kg/mm^2.

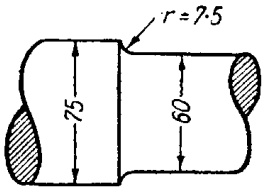
Determine the value of the endurance limit of the component in flexure: (a) for a symmetrical cycle; (b) for a cycle with the characteristic $r = 0{\cdot}2$; (c) for a cycle with the mean stress $\sigma_m = +8$ kg/mm^2; and (d) for a cycle with the minimum stress $\sigma_{\min} = -12$ kg/mm^2.

In solving the problem make use of the relationship obtained by considering the straight-line diagram of the limiting stresses within the co-ordinates $\sigma_a - \sigma_m$:

$$\sigma_a = \sigma_{-1}^b\left[1 - \frac{\sigma_m}{\sigma_y^b}\right].$$

Answer: (a) 20·3 kg/mm^2; (b) 28·8 kg/mm^2; (c) 32·4 kg/mm^2; (d) 23·0 kg/mm^2.

9.76. A stepped shaft with diameters D of 75 mm and d of 60 mm (see figure), made of a carbon steel, for which $\sigma_u = 75$ kg/mm^2, $\sigma_y^b = 45$ kg/mm^2 and $\sigma_{-1}^b = 35$ kg/mm^2, is bent by a moment varying from $+0{\cdot}5M_{\max}$ to $+M_{\max}$.



PROBLEM 9.76

Determine the maximum allowable value of M_{max} if the factor of safety k is 1·6. Follow the solution of Problem 9.73.

Answer: 440 kgm.

9.77. A helical valve spring made of steel for which $\tau_y^t = 80\,kg/mm^2$, $\tau_{-1}^t = 50\,kg/mm^2$ and $G = 8 \times 10^3\,kg/mm^2$, has 10 coils with a mean coil diameter of 40 mm and wire diameter of 4 mm. The initial tension on the spring is 12 kg. The working deflection of the spring varies from zero to 15 mm. The stress concentration factor for the helical spring at $D/d = 40/4 = 10$ may be taken to be 1·15.

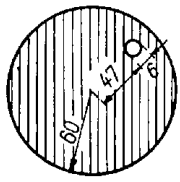
Determine the value of the limiting and actual shear stresses in the spring.

Answer: 70·2 kg/mm²; 30 kg/mm².

9.78. A round shaft, 120 mm in diameter, weakened by a round longitudinal hole 6 mm in diameter (see figure) is made of a carbon steel for which σ_u is 75 kg/mm², τ_y^t is 22 kg/mm² and τ_{-1}^t is 20 kg/mm². The shaft is being twisted by a torque varying from $+0{\cdot}4\,M_{max}$ to $+M_{max}$.

Determine the maximum allowable value of M_{max} if the factor of safety k is 1·8. The value of the theoretical stress concentration factor at the boundaries of the hole is to be taken as two. In calculating the polar second moment of area of the shaft the hole is not to be taken into account.

Answer: 2830 kgm.



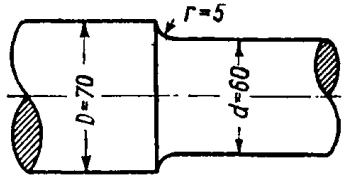
PROBLEM 9.78

9.79. A circular shaft 60 mm in diameter, having a fillet radius of 5 mm at the transition into the diameter of 70 mm (see figure), is made of a carbon steel for which σ_u is 80 kg/mm², σ_y^b is

55 kg/mm², σ^{b}_{-1} is 35 kg/mm², τ^{t}_{y} is 28 kg/mm² and τ^{t}_{-1} is 20 kg/mm².

The shaft is bent by a moment varying from $-M_{b\max}$ to $+M_{b\max} = 60$ kgm, and twisted by a torque varying from zero to $M_{t\max} = 180$ kgm; the bending moment and the torque reach their maximum and their minimum values respectively at the same time. The dynamic load coefficient for the variable component of the cycle of the normal and shear stresses is 2; the factor of safety is 1·8.

Check the strength of the shaft.



PROBLEM 9.79

Solution. The condition of limiting strength for the shaft subjected simultaneously to flexure and torsion may be expressed as

$$\sqrt{\left(\frac{\sigma_b}{[\sigma^{bc}_{rb}]}\right)^2 + \left(\frac{\tau_t}{[\tau^{tc}_{rt}]}\right)^2} \leq 1.$$

Here $[\sigma^{bc}_{rb}]$ and $[\tau^{tc}_{rt}]$ are the allowable stresses in flexure and torsion respectively, determined for the component by the degree of non-symmetry of the cycle of the normal and shear stresses.

We shall determine $[\sigma^{bc}_{rb}]$ and $[\tau^{tc}_{rt}]$.

The characteristic of the cycle in flexure is given by

$$r_b = \frac{\sigma_{min}}{\sigma_{max}} = \frac{M_{min}}{M_{max}} = \frac{-M_{max}}{M_{max}} = -1.$$

The value of the allowable stress in flexure (symmetrical cycle) will be determined by the formula

$$[\sigma^{bc}_{rb}] = [\sigma^{bc}_{-1}] = \frac{\sigma^{b}_{-1}}{k \times K_d \times \alpha_{af} \times \alpha_s}.$$

The value of the theoretical stress concentration factor in flexure of a shaft with a fillet radius may be found from Table 37*. For the ratio $r/d = 5/60 = 0{\cdot}0833$, linearly interpolating between the values of $\alpha_{tf} = 1{\cdot}75$ at $r/d = 0{\cdot}0625$ and $\alpha_{tf} = 1{\cdot}50$ at $r/d = 0{\cdot}125$, we have $\alpha_{tf} = 1{\cdot}67$. The value of the notch sensitivity is determined from the curve in Fig. 627*; at $\alpha_{tf} = 1{\cdot}67$ and $\sigma_u = 80$ kg/mm^2 we have $q = 0{\cdot}67$. Thus

$$\alpha_{af} = 1 + q(\alpha_{tf} - 1) = 1 + 0{\cdot}67(1{\cdot}67 - 1) = 1{\cdot}45.$$

The value of the scale factor is determined from the curve 2 in Fig. 629*. At $d = 60$ mm we have $\alpha_s = 1{\cdot}46$. Since $k = 1{\cdot}8$ and $K_d = 2$,

$$[\sigma_{rb}^{bc}] = \frac{3500}{1{\cdot}8 \times 2 \times 1{\cdot}45 \times 1{\cdot}46} = 459 \text{ kg/cm}^2.$$

The characteristic of the cycle in torsion is $r_t = \tau_{min}/\tau_{max} = M_{tmin}/M_{tmax} = 0$. The value of the allowable stress for a symmetrical cycle in torsion is determined by the formula

$$[\tau_{-1}^{tc}] = \frac{\tau_{-1}^{t}}{k \times K_d \times \alpha_{af} \times \alpha_s}.$$

The value of the theoretical stress concentration factor in torsion for a shaft with a fillet radius will be found from Table 37*. Interpolating between the values of $\alpha_{tf} = 1{\cdot}2$ (at $r/d = 0{\cdot}1$) and $\alpha_{tf} = 1{\cdot}8$ (at $r/d = 0{\cdot}02$), we find that $\alpha_{tf} \approx 1{\cdot}3$ for the ratio $r/d = 0{\cdot}0833$. The value of notch sensitivity q at $\alpha_{tf} = 1{\cdot}3$ and $\sigma_u = 80$ kg/mm^2 is 0·55. Therefore

$$\alpha_{af} = 1 + 0{\cdot}55(1{\cdot}3 - 1) = 1{\cdot}17.$$

The value of the scale factor is the same as in flexure: $\alpha_s = 1{\cdot}46$. Thus, the allowable stress for the symmetrical cycle in torsion is

$$[\tau_{-1}^{tc}] = \frac{2000}{1{\cdot}8 \times 2 \times 1{\cdot}17 \times 1{\cdot}46} = 325 \text{ kg/cm}^2.$$

* See Appendix B, Section 6.

The allowable stress for the constant component of the cycle in torsion is

$$[\tau_{+1}^{tc}] = \frac{\tau_{+1}^{tc}}{k} = \frac{\tau_y^t}{k} = \frac{2800}{1{\cdot}8} = 1555 \text{ kg/cm}^2.$$

The value of the allowable stress on torsion of the component, for the cycle with the characteristic $r = 0$, is determined by the formula

$$[\tau_{rt}^{tc}] = [\tau_0^{tc}] = \frac{2\,[\tau_{-1}^{tc}]\,[\tau_{+1}^{tc}]}{(1+r)\,[\tau_{-1}^{tc}] + (1-r)\,[\tau_{+1}^{tc}]}$$

$$= \frac{2 \times 1555 \times 325}{(1+0)325 + (1-0)1555} = 536 \text{ kg/cm}^2.$$

The maximum actual values of the stresses σ_b and τ_t, in flexure and torsion respectively, are:

$$\sigma_b = \frac{M_b}{W_b} = \frac{32M_b}{\pi d^3} = \frac{32 \times 6000}{3{\cdot}14 \times 6^3} = 283 \text{ kg/cm}^2$$

and

$$\tau_t = \frac{M_t}{W_p} = \frac{16M_t}{\pi d^3} = \frac{16 \times 18{,}000}{3{\cdot}14 \times 6^3} = 424 \text{ kg/cm}^2.$$

Substituting for the limiting strength condition the values of σ_b and τ_t, and also the values (found earlier) of $[\sigma_{rb}^{bc}]$ and $[\tau_{rt}^{tc}]$ we have

$$\sqrt{\left(\frac{283}{459}\right)^2 + \left(\frac{424}{536}\right)^2} = 1{\cdot}003 \approx 1.$$

The strength of the shaft is adequate.

9.80. A shaft 120 mm in diameter, made of an alloy steel for which $\sigma_u = 95$ kg/mm^2, $\sigma_{-1}^b = 42$ kg/mm^2, $\tau_y^t = 40$ kg/mm^2 and $\tau_{-1}^t = 24$ kg/mm^2, has a fillet radius of 10 mm at the transition to a diameter of 100 mm. The shaft is being twisted by a torque varying from 600to 1800 kgm, and bent by a bending moment varying from $-M_{max}$ to $+M_{max}$. Assuming that the factor of

safety is 1·7, determine the maximum allowable value of the bending moment.

Answer: 603 kg m.

9.81. The pin, 75 mm in diameter, of a crankshaft has a small hole for lubrication. The material of the pin is a carbon steel for which $\sigma_u = 65\,\text{kg/mm}^2$, $\sigma_y^b = 42\,\text{kg/mm}^2$, $\sigma_{-1}^b = 28\,\text{kg/mm}^2$, $\tau_y^t = 22\,\text{kg/mm}^2$ and $\tau_{-1}^t = 16\,\text{kg/mm}^2$.

The pin is being subjected to a bending moment varying from $-0{\cdot}8M_{b\max}$ to $+M_{b\max}$, and a torque varying from $-0{\cdot}1M_{t\max}$ to $+M_{t\max} = 2M_{b\max}$. The actual stress concentration factor, on account of the presence of the oil hole, is to be assumed 2·6 in flexure, and 2·3 in torsion. The dynamic factor of the variable load component in flexure, and in torsion, is 2; the factor of safety, k, is 1·8.

Determine the maximum allowable values of $M_{b\max}$ and $M_{t\max}$.

Answer: $M_{b\max} = 93{\cdot}4\,\text{kg m}$; $M_{t\max} = 186{\cdot}9\,\text{kg m}$.

40. CREEP

9.82. The bolts of a flanged joint of a turbine casing have diameter d of 20 mm and length l of 80 mm. The initial elastic tension P_0 of each bolt is 4 t. The joint works at a temperature (T) of 450°C.

Taking into account the creep of the material of the bolts and flanges, determine the tension in the bolts after 5000 hr from the start of the turbine. Neglect the stage of primary creep.

At a temperature of 450°C the modulus of elasticity for the material of the bolts is $1{\cdot}6 \times 10^6\,\text{kg/cm}^2$. The rate of a steady (established) creep strain may be calculated by the formula

$$\frac{d\varepsilon_{bc}}{dt} = k_1 \sigma_b^n,$$

whereby, at $T = 450°\text{C}$:

$$k_1 = 2 \times 10^{-22} \left(\frac{\text{cm}^2}{\text{kg}}\right)^n \frac{1}{\text{hr}} \qquad \text{and} \qquad n = 5{\cdot}5.$$

The stiffness m for the flanges in elastic deformation is 2×10^6 kg/cm^2; the rate of steady creep deformation may be calculated by the formula $\mathrm{d}\Delta_{\mathrm{flc}}/\mathrm{d}t = k_2 P_t^n$, where P_t is the tension in the bolt at a time instant t, k_2 is 2×10^{-27}cm^{2n}/kg^{n}hr and n is 5·5 (at $T = 450$°C).

Solution. The total deformation of the bolt and flanges, which before the commencement of creep is

$$\Delta l_0 = \Delta l_{\mathrm{be}} + \Delta l_{\mathrm{fle}} = \frac{P_0 l}{E_{\mathrm{T}} F} + \frac{P_0}{m}.$$

does not vary with time, and remains equal to Δl_0. During creep the total deformation of the bolt and flanges is made up of the elastic deformation of the bolt (Δl_{be}) and flanges (Δl_{fle}), and the creep deformation of the bolt (Δl_{bc}) and flanges (Δl_{fle}):

$$\Delta l_{\mathrm{be}} + \Delta l_{\mathrm{fle}} + \Delta l_{\mathrm{bc}} + \Delta l_{\mathrm{fle}} = \frac{P_t l}{E_{\mathrm{T}} F} + \frac{P_t}{m} + \Delta l_{\mathrm{bc}} + \Delta l_{\mathrm{fl}} = \Delta l_0.$$

Differentiating this equation with respect to t, we have

$$\left(\frac{l}{E_{\mathrm{T}} F} + \frac{1}{m}\right) \frac{\mathrm{d}P_t}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \Delta l_{\mathrm{bc}} + \frac{\mathrm{d}}{\mathrm{d}t} \Delta l_{\mathrm{flc}} = 0.$$

Neglecting the stage of primary creep and substituting

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta l_{\mathrm{bc}} = l \frac{\mathrm{d}\varepsilon_{\mathrm{bc}}}{\mathrm{d}t} \quad \text{by} \quad l k_1 \sigma^n = k_1 \left(\frac{P_t}{F}\right)^n l,$$

$$\text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \Delta l_{\mathrm{flc}} \quad \text{by} \quad k_2 P_t^n$$

we obtain the following differential equation in P_t

$$\left(\frac{l}{E_{\mathrm{T}} F} + \frac{1}{m}\right) \frac{\mathrm{d}P_t}{\mathrm{d}t} + \left(k_1 \frac{l}{F^n} + k_2\right) P_t^n = 0,$$

or

$$\frac{\mathrm{d}P_t}{P_t^n} = -\frac{k_1 \dfrac{l}{F^n} + k_2}{\dfrac{l}{E_{\mathrm{T}} F} + \dfrac{1}{m}} \mathrm{d}t = -A\,\mathrm{d}t.$$

Solving this equation we find that

$$\frac{1}{(n-1)\,P_t^{n-1}} = At + C,$$

where C is a constant of integration. Since at $t = 0$ P_t is P_0,

$$C = \frac{1}{(n-1)P_0^{n-1}} \quad \text{and} \quad P_t = \frac{P_0}{[1 + (n-1)\,At\,P_0^{n-1}]^{\frac{1}{n-1}}}.$$

Inserting here the numerical data,

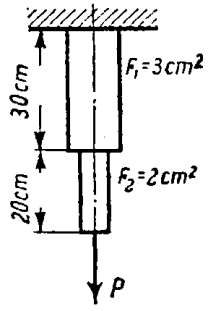
$$A = \frac{k_1 \dfrac{l}{F^n} + k_2}{\dfrac{l}{E_T F} + \dfrac{1}{m}} = \frac{\left(2 \times 10^{-22} \times \dfrac{8}{3{\cdot}14^{5{\cdot}5}}\right) + (2 \times 10^{-27})}{\dfrac{8}{1{\cdot}6 \times 10^6 \times 3{\cdot}14} + \dfrac{1}{2 \times 10^6}}$$

$$= 3{\cdot}911 \times 10^{-21} \frac{1}{\mathrm{kg}^{n-1}} \times \frac{1}{\mathrm{hr}}$$

and

$$P_t = \frac{4000}{[1 + (4{\cdot}5 \times 3{\cdot}911 \times 10^{-21} \times 4000^{4{\cdot}5} \times 5000)]^{\frac{1}{4{\cdot}5}}} = 3285\ \mathrm{kg}.$$

9.83. A stepped steel bar (see figure) is subjected to a tensile force P. Determine the maximum allowable value of this force at



PROBLEM 9.83

a temperature of 650°C, proceeding from the condition that the maximum rate of the displacement of the lower end of the bar is

not to exceed 10^{-5} cm/hr. The rate of steady creep for the material of the bar may be calculated by the formula

$$v = \frac{d\varepsilon_c}{dt} = k\sigma^n;$$

at a temperature of 650°C k is $2{\cdot}5 \times 10^{-15}(\text{cm}^2/\text{kg})^n\,\text{hr}^{-1}$, and n is 3.

Answer: 1035 kg.

9.84. A steel rod having a cross-sectional area of 2 cm² is subjected to a tensile force of 1·8 t. Assuming that the deformation of the rod during the period of primary creep is approximately 60% of its elastic deformation, determine the service period for the rod at a temperature of 500°C. At this temperature the modulus of elasticity, E, of the material of the rod is $1{\cdot}6 \times 10^6$ kg/cm², and the rate of steady creep v_c is $d\varepsilon_c/dt = a \sinh \sigma/b$, where a is 2×10^{-7} hr^{-1} and b is 240 kg/cm². The allowable value of the strain is 1%.

Answer: 2140 hr.

9.85. A steel rod having a length of 60 cm and cross-sectional area of 4 cm² is subjected to a tensile force of 2 t at a temperature of 600°C. Determine the limiting service period for the rod, if its elongation is not to exceed 2 mm, and the maximum normal stress is not to exceed a half of the maximum long-term stress. For the rate of steady creep the following may be assumed: $v_c = d\varepsilon/dt = k\sigma^n$, where k is $10{\cdot}6 \times 10^{-25}\,(\text{cm}^2/\text{kg})^n\,\text{hr}^{-1}$ and n is 7. The value of the maximum long-term stress can be calculated by the formula $\sigma_{ut} = 28/t^{0{\cdot}12}$ (kg/mm²). At a temperature of 600°C, E is $1{\cdot}5 \times 10^6$ kg/cm². The strain in the stage of primary creep is to be assumed approximately equal to the strain in the elastic stage.

Answer: 5325 hr.

9.86. A circular steel shaft is being twisted by a torque M_t of 300 kgm. Determine the maximum shear stress τ_{max} in the cross-section of the shaft, and the angle of twist Φ per unit length at room temperature, and also at a temperature T of 540°C after 2000 hr from the start of loading. Compare the results of the calculation (τ_{max} and Φ) for a solid shaft, having a diameter d_0

of 56 mm, with those for a hollow shaft having an external diameter (d_2) of 61 mm and internal diameter (d_1) of 42 mm. The stage of primary creep should not be taken into consideration. Neglect the rate of elastic deformation.

The material of the shaft is a carbon steel for which G at room temperature is 8×10^5 kg/cm^2; G_T at a temperature of 540°C is 6×10^5 kg/cm^2. The rate of steady creep is given by $v_c = k\tau^n$; at 540°C k is 3×10^{-20} (cm^2/kg)^n hr^{-1} and n is 5.

Solution. At room temperature the maximum shear stress in the solid shaft

$$\tau_{1\max} = \frac{M_t}{W_{p1}} = \frac{16M_t}{\pi d_0^3} = \frac{16 \times 30{,}000}{3{\cdot}14 \times 5{\cdot}6^3} = 870 \text{ kg/cm}^2,$$

and that in the hollow shaft

$$\tau_{2\max} = \frac{M_t}{W_{p2}} = \frac{16M_t\, d_2}{\pi(d_2^4 - d_1^4)} = \frac{16 \times 30{,}000 \times 6{\cdot}1}{3{\cdot}14(6{\cdot}1^4 - 4{\cdot}2^4)}$$
$$= 868 \text{ kg/cm}^2 \approx \tau_{1\max}.$$

The angle of twist per unit length of the solid shaft

$$\Phi_1 = \frac{M_t}{GJ_{p1}} = \frac{32 \times 30{,}000}{8 \times 10^5 \times 3{\cdot}14 \times 5{\cdot}6^4} = 0{\cdot}000388 \text{ cm}^{-1},$$

and that of the hollow shaft

$$\Phi_2 = \frac{M_t}{GJ_{p2}} = \frac{32M_t}{G\pi(d_2^4 - d_1^4)} = \frac{32 \times 30{,}000}{8 \times 10^5 \times 3{\cdot}14(6{\cdot}1^4 - 4{\cdot}2^4)}$$
$$= 0{\cdot}000356 \text{ cm}^{-1} < \Phi_1.$$

During creep the maximum shear stress, and the angle of twist per unit length are calculated by the formulae

$$\tau_{\max} = \frac{M_t}{J_{pc}}\left(\frac{d}{2}\right)^{\frac{1}{n}} \quad \text{and} \quad \Phi = \frac{M_t}{G_T J_p} + k\left(\frac{M_t}{J_{pc}}\right)^n t.$$

For the solid shaft

$$J_{pc} = 2\pi \int_0^{\frac{d_0}{2}} \varrho^{\frac{2n+1}{n}}\, d\varrho = \frac{2\pi n}{3n+1}\left(\frac{d_0}{2}\right)^{\frac{3n+1}{n}}$$
$$= \frac{2 \times 3{\cdot}14 \times 5}{16} \times 2{\cdot}8^{3{\cdot}2} = 52{\cdot}96 \text{ cm}^{3{\cdot}2},$$

and for the hollow shaft

$$J_{pc} = 2\pi \int_{\frac{d_1}{2}}^{\frac{d_2}{2}} \varrho^{\frac{2n+1}{n}} d\varrho = \frac{2\pi n}{3n+1}\left[\left(\frac{d_2}{2}\right)^{\frac{3n+1}{n}} - \left(\frac{d_1}{2}\right)^{\frac{3n+1}{n}}\right]$$

$$= \frac{2 \times 3{\cdot}14 \times 5}{16}(3{\cdot}05^{3{\cdot}2} - 2{\cdot}1^{3{\cdot}2}) = 48{\cdot}54\,\text{cm}^{3\,2}.$$

Therefore during creep the maximum shear stress in the solid shaft

$$\tau_{1\max} = \frac{30{,}000 \times 2{\cdot}8^{\frac{1}{5}}}{52{\cdot}96} = 696\ \text{kg/cm}^2,$$

and that in the hollow shaft

$$\tau_{2\max} = \frac{30{,}000 \times 3{\cdot}05^{\frac{1}{5}}}{48{\cdot}54} = 773\ \text{kg/cm}^2 > \tau_{1\max}.$$

The angle of twist per unit length in the solid shaft

$$\Phi_1 = \frac{30{,}000 \times 32}{6 \times 10^5 \times 3{\cdot}14 \times 5{\cdot}6^4} + \left[3 \times 10^{-20}\left(\frac{30{,}000}{52{\cdot}96}\right)^5 \times 2000\right]$$

$$= 0{\cdot}00052 + 0{\cdot}00350 = 0{\cdot}00402\ \text{cm}^{-1},$$

and that in the hollow shaft

$$\Phi_2 = \frac{30{,}000 \times 32}{6 \times 10^5 \times 3{\cdot}14(6{\cdot}1^4 - 4{\cdot}2^4)} + \left[3 \times 10^{-20} \times \left(\frac{30{,}000}{48{\cdot}54}\right)^5 \times 2000\right]$$

$$= 0{\cdot}00047 + 0{\cdot}00541 = 0{\cdot}00588 > \Phi_1.$$

Thus, at room temperature the maximum shear stresses in the solid and hollow shafts are nearly equal, whereas the angle of twist of the hollow shaft is approximately 10% smaller than that of the

solid shaft. Therefore the hollow shaft is superior to the solid one. During creep the maximum shear stresses in both shafts are reduced; in the solid shaft they fall more noticeably than in the hollow shaft. The angles of twist of the shafts increase sharply; thereby the angle of twist of the hollow shaft is about 1·5 times greater than that of the solid shaft. Therefore in creep the solid shaft is superior to the hollow one.

9.87. A circular steel shaft is being twisted by a torque (M) of 600 kgm. Neglecting the stage of primary creep, determine the required diameter of the shaft—so that at a temperature of 500°C the maximum rate of steady creep would not exceed 10^{-7} hr^{-1}. The material of the shaft is a steel for which at 500°C the creep rate is given by $v_c = k\tau^n$, whereby k is $5{\cdot}6 \times 10^{-20}$ (cm^2/kg)^n hr^{-1} and n is 4·8.

Answer: $d = 44{\cdot}5$ mm ≈ 45 mm.

9.88. A helical spring having a mean coil diameter of 36 mm and wire diameter of 4 mm is compressed by a force of 3·5 kg. The spring works at a temperature of 500°C.

Determine the number of coils of the spring necessary to ensure that during the shortening of the spring the rate of deformation would not exceed 10^{-6} cm/hr. The rate of the steady creep strain for the material of the spring is given by $v_c = k\tau^n$, whereby at 500°C, k is 4×10^{-20} (cm^2/kg)^n hr^{-1} and n is 4. The stage of primary creep should not be taken into consideration.

Answer: $8{\cdot}95 \approx 9$ coils.

9.89. A cantilever 60 cm long having a circular cross-section 40 mm in diameter, and being loaded by a uniformly distributed load of intensity q, works at a temperature (T) of 600°C. The rate of steady creep is given by $v_c = k\sigma^n$; at 600°C, k is $0{\cdot}25 \times 10^{-12}$ (cm^2/kg)^n hr^{-1} and n is 3.

Neglecting the stage of primary creep, determine the maximum allowable value of the load q—so that the deflection after 5000 hr from the initial loading would not exceed 5 mm, and the maximum normal stress would not exceed the allowable long-term limiting stress σ_{ut} which is 200 kg/cm^2 for $t = 5000$ hr. Take the factor of safety k_t as 1·8.

Solution. The condition of the limiting normal stresses in the given case may be expressed as

$$\sigma_{\max} = \frac{M_{\max}}{J_{zn}} r^{\frac{1}{n}} \leq [\sigma_{ut}] = \frac{\sigma_{ut}}{k_t}.$$

Here r is the radius of the cross-section of the beam, z is the neutral axis of the cross-section, and

$$J_{zn} = \int_F y^{\frac{n+1}{n}} \, dF = 2 \int_0^r y^{\frac{n+1}{n}} \, 2\sqrt{r^2 - y^2} \, dy = m r^{\frac{3n+1}{n}}.$$

The values of factor m, corresponding to the different values of index n, are given in the table below.

n	m
1	0·785
1·5	0·918
2	1·009
2·5	1·070
3	1·104
4	1·144
5	1·168
6	1·189
8	1·230
10	1·270

Inserting the numerical values we obtain

$$J_{zn} = 1{\cdot}104 \times 2^{\frac{10}{3}} = 11{\cdot}13 \text{ cm}^{\frac{10}{3}}$$

and

$$\sigma_{\max} = \frac{q l^2}{2 J_{zn}} r^{\frac{1}{n}} = \frac{q \times 10^2 \times 2^{\frac{1}{3}}}{2 \times 11{\cdot}13}$$

$$= 203{\cdot}8q \leq \frac{\sigma_{ut}}{k_t} = \frac{200}{1{\cdot}8} = 111{\cdot}1 \text{ kg/cm}^2$$

whence

$$q \leq \frac{111{\cdot}1}{203{\cdot}8} = 0{\cdot}545 \text{ kg/cm}.$$

The elastic deflection of the free end of the cantilever is

$$f_e = \frac{ql^4}{8E_T J_z} = \frac{q \times 60^4 \times 4}{8 \times 1{\cdot}5 \times 10^6 \times 3{\cdot}14 \times 2^4} = 0{\cdot}086q.$$

For the calculation of the deflection of the same point during creep we use the formula (see *Strength of Materials* by N. M. Belyayev, 1954 Edition; Chapter 29, Fig. 671*)

$$f_c = \frac{kq^n l^{2n+2} t}{2^n (2n+2) J_{zc}^n} = \frac{0{\cdot}25 \times 10^{-12} \times q^3 \times 60^8 \times 5000}{2^3[(2 \times 3) + 2] \times 11{\cdot}13^3} = 23{\cdot}8q^3.$$

It is stipulated that the total deflection of the beam must not exceed 5 mm, i.e.

$$f_e + f_c = 0{\cdot}086q + 23{\cdot}8q^3 \leq [f] = 0{\cdot}5 \text{ cm}.$$

Solving this equation for q we find that

$$q \leq 0{\cdot}272 \text{ kg/cm}.$$

The load on the beam must not exceed 0·272 kg/cm.

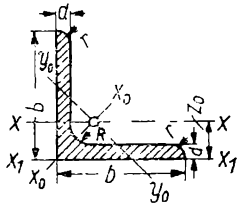
9.90. A cantilever 30 cm long, loaded at the free end by a concentrated load of 500 kg, works at a temperature of 540°C. The material of the beam is a carbon steel with a modulus of elasticity, E_T, of $1{\cdot}6 \times 10^6$ kg/cm². The rate of steady creep is given by $v = k\sigma^n$; at 540°C, k is $1{\cdot}4 \times 10^{-16}$ (cm²/kg)^n hr^{-1} and n is 3·9.

Neglecting the stage of primary creep determine the maximum normal stress in the critical cross-section of the beam, and its maximum deflection after 4000 hr from the loading. Investigate two cross-sections with equal section moduli in flexure: one rectangular with a depth of 80 mm and width of 29 mm (the depth is parallel to the plane of action of the load), and the other circular 68 mm in diameter. When calculating the beam of circular cross-section make use of the instructions given in Problem 9.89.

Answer: In the beam of rectangular cross-section: $\sigma_{max} = 365$ kg/cm², $f_{max} = 0.232$ cm; in the beam of circular cross-section: $\sigma_{max} = 335$ kg/cm², $f_{max} = 0{\cdot}203$ cm. In creep the circular cross-section is superior to the rectangular one.

* Appendix B, Section 6.

APPENDIXES



APPENDIX A
Rolled Steel Sections (OST 10014-39 to 10017-39)

TABLE 1. EQUAL ANGLES

Sec. No.	Wt./m in kg	Dimensions in mm				Area F of section in cm^2	xx Axis		x_0x_0 Axis		y_0y_0 Axis		x_1x_1 Axis	Distance z_0 of centroid in cm
		b	d	R	r		J_x, cm^4	i_x, cm	$J_{x_{0max}}$, cm^4	$i_{x_{0max}}$, cm	$J_{y_{0min}}$, cm^4	$i_{y_{0min}}$, cm	J_{x_1}, cm^4	
2	0·89	20	3	3·5	1·2	1·13	0·40	0·60	0·63	0·75	0·17	0·39	0·81	0·60
	1·15		4			1·46	0·49	0·58	0·78	0·73	0·21	0·38	1·09	0·64
2·5	1·12	25	3	3·5	1·2	1·43	0·80	0·75	1·28	0·95	0·32	0·47	1·56	0·73
	1·46		4			1·86	1·02	0·74	1·63	0·94	0·41	0·47	2·12	0·77
3	1·78	30	4	4·5	1·5	2·27	1·82	0·90	2·90	1·13	0·75	0·58	3·62	0·89
	2·18		5			2·78	2·20	0·89	3·47	1·12	0·93	0·58	4·58	0·93
3·5	2·10	35	4	4·5	1·5	2·67	3·02	1·06	4·47	1·34	1·27	0·69	5·74	1·01
	2·57		5			3·28	3·61	1·05	5·71	1·32	1·51	0·68	7·23	1·05
4	2·42	40	4	5·0	1·7	3·08	4·60	1·22	7·26	1·53	1·93	0·79	8·53	1·13
	2·97		5			3·79	5·54	1·21	8·75	1·52	2·34	0·78	10·7	1·17
	3·52		6			4·48	6·41	1·20	10·1	1·50	2·70	0·78	13·0	1·21

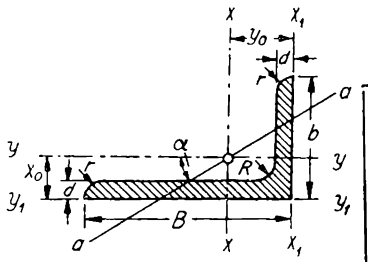
4.5	2·73	45	4	5·0	1·7	3·48	6·60	1·38	10·5	1·74	2·68	0·88	12·1	1·26
	3·37		5			4·29	8·00	1·37	12·7	1·72	3·26	0·87	15·3	1·30
	3·99		6			5·08	9·29	1·35	14·8	1·71	3·78	0·86	18·4	1·34
5	3·77	50	5	5·5	1·8	4·80	11·2	1·53	17·8	1·92	4·61	0·98	20·9	1·42
	4·47		6			5·69	13·1	1·52	20·7	1·91	5·39	0·97	25·2	1·46
6	5·47	60	5	6·5	2·2	5·82	19·9	1·85	31·4	2·32	8·29	1·19	35·9	1·66
	5·42		6			6·91	23·3	1·84	36·8	2·31	9·76	1·19	43·3	1·70
	7·09		8			9·03	29·6	1·81	46·8	2·28	12·4	1·17	58·2	1·78
6·5	5·93	65	6	8·0	2·7	7·55	29·8	1·98	47·2	2·50	12·3	1·28	54·8	1·82
	7·75		8			9·87	38·1	1·96	60·3	2·48	15·8	1·27	73·7	1·90
	9·51		10			12·1	45·4	1·94	72	2·44	18·8	1·25	92·9	1·98
7·5	6·89	75	6	9·0	3·0	8·78	46·7	2·31	73·3	2·89	20·1	1·51	83·9	2·06
	9·03		8			11·5	60·1	2·28	94·9	2·87	25·3	1·48	113	2·14
	11·1		10			14·1	72·2	2·26	114	2·84	30·3	1·47	142	2·22
	13·1		12			16·7	83·3	2·24	132	2·81	34·9	1·45	172	2·30
8	7·36	80	6	9·0	3·0	9·38	57·0	2·47	90	3·11	23·5	1·58	102	2·19
	9·66		8			12·3	73·3	2·44	116	3·07	30·3	1·57	137	2·27
	11·9		10			15·1	88·4	2·42	140	3·05	36·5	1·56	172	2·35
9	11·0	90	8	11	3·7	14·0	106	2·76	168	3·46	43·6	1·77	194	2·51
	13·5		10			17·2	128	2·74	204	3·44	53·1	1·76	244	2·59
	16·0		12			20·4	149	2·71	237	3·41	61·7	1·75	294	2·67
	18·4		14			23·4	169	2·69	267	3·38	71·9	1·75	345	2·74

TABLE 1—*contd.*

Sec. No.	Wt./m in kg	Dimension in mm				Area F of section in cm^2	xx Axis		x_0x_0 Axis		y_0y_0 Axis		x_1x_1 Axis	Distance z_0 of centroid in cm
		b	d	R	r		J_x, cm^4	i_x, cm	$J_{x_{0max}}$, cm^4	$i_{x_{0max}}$, cm	$J_{y_{0min}}$, cm^4	$i_{y_{0min}}$, cm	J_{x1}, cm^4	
10	12·3	100	8	12	4·0	15·6	147	3·07	233	3·87	61·0	1·98	265	2·75
	15·1		10			19·2	179	3·05	284	3·85	74·9	1·97	333	2·83
	17·9		12			22·8	209	3·03	331	3·81	87·6	1·96	402	2·91
	20·6		14			26·3	237	3·01	375	3·78	99·1	1·95	472	2·99
	23·3		16			29·7	265	2·99	416	3·75	113	1·95	542	3·06
12	18·3	120	10	13	4·3	23·3	316	3·68	503	4·64	130	2·36	575	3·33
	21·7		12			27·6	371	3·66	590	4·62	153	2·35	693	3·41
	25·1		14			31·9	423	3·64	671	4·59	174	2·34	811	3·49
	28·4		16			36·1	474	3·62	749	4·56	199	2·34	931	3·56
	31·6		18			40·3	520	3·60	822	4·54	219	2·33	1052	3·64
13	19·8	130	10	13	4·3	25·3	406	4·01	646	5·05	166	2·56	730	3·58
	23·6		12			30·0	477	3·99	759	5·03	196	2·55	880	3·66
	27·3		14			34·7	545	3·96	866	5·00	223	2·54	1030	3·74
	30·9		16			39·3	608	3·93	967	4·96	249	2·52	1182	3·82

15	27·4	150	12	15	5·0	34·9	745	4·62	1186	5·83	305	2·95	1347	4·15
	31·7		14			40·4	857	4·60	1358	5·80	356	2·97	1577	4·22
	36·0		16			45·8	961	4·58	1522	5·77	399	2·95	1808	4·30
	40·1		18			51·1	1060	4·56	1679	5·73	440	2·94	2041	4·38
	44·3		20			56·4	1154	4·52	1830	5·70	478	2·91	2275	4·46
18	38·3	180	14	15	5·0	48·8	1515	5·57	2405	7·03	625	3·58	2721	4·97
	43·5		16			55·4	1704	5·55	2705	7·00	703	3·56	3117	5·05
	48·6		18			61·9	1885	5·52	2994	6·94	775	3·54	3515	5·13
20	48·7	200	16	18	6·0	62·0	2355	6·17	3755	7·79	954	3·93	4264	5·55
	54·4		18			69·3	2619	6·15	4165	7·77	1074	3·93	4808	5·62
	60·1		20			76·5	2868	6·13	4560	7·74	1175	3·92	5355	5·70
	71·3		24			90·8	3349	6·07	5313	7·66	1384	3·90	6456	5·85
	88·3		30			111·5	3999	5·96	6373	7·49	1685	3·87	8093	6·03
22	53·7	220	16	21	7·0	68·4	3168	6·80	5046	8·59	1307	4·37	5661	6·04
	66·4		20			84·5	3859	6·75	6120	8·51	1592	4·34	7003	6·20
	78·8		24			100·4	4514	6·71	7148	8·44	1870	4·32	8560	6·35
	91·0		28			115·9	5135	6·66	8130	8·37	2141	4·30	100,29	6·50
23	82·6	230	24	20	7·0	105·3	5207	7·03	8266	8·86	2144	4·51	9780	6·59

Table 2. Unequal Angles



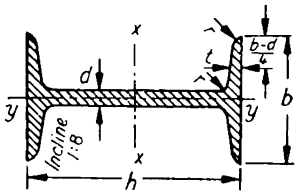
Sec. No.	Dimensions in mm					Area F of section in cm²	Weight/m in kg	Axes reference values											
								xx Axis		yy Axis		x_1x_1 Axis		y_1y_1 Axis		uu Axis		
	B	b	d	R	r			J_x, cm⁴	i_x, cm	J_y, cm⁴	i_y, cm	J_{x_1}, cm⁴	Distance y_0 of centroid in cm	$J_{y\min}$, cm⁴	Distance x_0 of centroid in cm	$J_{u\min}$, cm⁴	$i_{\min}$, cm	tan α
3/2	30	20	3	3·5	1·2	1·43	1·12	1·27	0·94	0·45	0·56	2·69	1.00	0·82	0·51	0·26	0·43	0·430
			4			1·86	1·46	1·61	0·93	0·56	0·55	3·61	1·04	1·12	0·55	0·34	0·43	0·422
3·5/2	35	20	4	3·5	1·2	2·06	1·62	2·48	1·10	0·53	0·53	5·62	1·25	1·11	0·51	0·37	0·42	0·320
			5			2·52	1·98	2·98	1·09	0·70	0·53	7·15	1·29	1·45	0·55	0·45	0·42	0·310
4·5/3	45	30	4	5·0	1·7	2·88	2·26	5·81	1·42	2·06	0·84	12·1	1·48	3·65	0·72	1·21	0·65	0·432
			6			4·18	3·28	8·15	1·40	2·85	0·83	18·3	1·56	5·65	0·82	1·69	0·64	0·423
6/4	60	40	5	7·0	2·3	4·83	3·79	17·4	1·90	6·19	1·13	35·8	1·95	10·8	0·97	3·62	0·87	0·432
			6			5·72	4·49	20·3	1·88	7·20	1·12	43·1	2·0	13·1	1·01	4·20	0·86	0·430
			8			7·44	5·84	25·8	1·86	9·04	1·10	57·9	2·08	17·9	1·09	5·39	0·85	0·420

7·5/5	75	50	5	8·0	2·7	6·11	4·80	34·9	2·39	12·5	1·43	69·9	2·39	20·9	1·17	7·24	1·09	0·435
			6			7·25	5·69	41·0	2·37	14·6	1·42	84·0	2·44	25·3	1·21	8·48	1·08	0·434
			8			9·47	7·43	52·4	2·35	18·6	1·40	113	2·52	34·3	1·29	10·9	1·07	0·428
			10			11·6	9·11	63·0	2·33	22·1	1·38	141	2·60	43·7	1·36	13·2	1·07	0·423
8/5·5	80	55	6	8·0	2·7	7·85	6·16	50·6	2·53	19·6	1·58	102	2·56	33·4	1·33	11·1	1·19	0·462
			8			10·3	8·06	64·9	2·51	24·9	1·56	136	2·64	45·3	1·41	14·3	1·19	0·458
			10			12·6	9·90	78·2	2·49	29·8	1·54	171	2·72	57·5	1·48	17·4	1·18	0·452
9/6	90	60	6	9	3	8·78	6·90	72·4	2·87	26·0	1·72	145	2·88	43·4	1·41	14·7	1·30	0·437
			8			11·5	9·08	93·2	2·85	33·2	1·70	194	2·96	58·6	1·49	19·3	1·29	0·433
			10			14·1	11·10	113·0	2·82	39·8	1·68	244	3·05	74·4	1·56	23·5	1·29	0·429
10/7·5	100	75	8	10	3·3	13·5	10·6	135	3·16	65·0	2·19	266	3·11	113	1·88	35·4	1·62	0·548
			10			16·7	13·1	163	3·13	78·5	2·17	333	3·20	143	1·96	42·6	1·60	0·545
			12			19·7	15·5	190	3·11	91·1	2·15	402	3·27	173	2·04	49·8	1·59	0·541
12/8	120	80	8	11	3·7	15·6	12·2	229	3·83	82·3	2·30	459	3·85	137	1·88	47·6	1·75	0·438
			10			19·2	15·1	279	3·81	99·6	2·27	575	3·93	178	1·96	57·7	1·73	0·436
			12			22·8	17·9	326	3·79	116	2·25	692	4·01	210	2·04	66·6	1·71	0·432
13/9	130	90	8	12	4·0	17·2	13·5	297	4·15	118	2·62	583	4·08	195	2·11	66·5	1·97	0·471
			10			21·3	16·7	362	4·12	143	2·59	731	4·16	245	2·19	81·4	1·95	0·469
			12			25·2	19·8	424	4·10	167	2·57	879	4·25	297	2·27	94·8	1·94	0·467
			14			29·1	22·8	484	4·08	189	2·55	1028	4·33	349	2·35	109	1·93	0·464

TABLE 2—*contd.*

Sec. No.	Dimensions in mm					Area F of section in cm²	Weight/m in kg	Axes reference values											
								xx Axis		yy Axis		x_1x_1 Axis		y_1y_1 Axis		uu Axis		
	B	b	d	R	r			J_x, cm⁴	i_x, cm	J_y, cm⁴	i_y, cm	J_{x_1}, cm⁴	Distance y_0 of centroid in cm	$J_{y_{min}}$, cm⁴	Distance x_0 of centroid in cm	J_{umin}, cm⁴	i_{min}, cm	tan α
15/10	150	100	10	13	4·3	24·3	19·1	557	4·78	201	2·87	1120	4·81	335	2·35	115	2·18	0·439
			12			28·8	22·6	655	4·76	235	2·85	1347	4·90	405	2·43	137	2·18	0·437
			14			33·3	26·2	749	4·73	267	2·83	1575	4·98	476	2·51	156	2·16	0·434
			16			37·7	29·6	839	4·71	297	2·81	1804	5·06	549	2·59	175	2·15	0·430
18/12	180	120	12	14	4·7	34·9	27·4	1156	5·75	417	3·46	2325	5·79	695	2·82	240	2·62	0·440
			14			40·4	31·7	1326	5·73	476	3·44	2717	5·87	816	2·90	279	2·62	0·438
			16			45·8	35·9	1490	5·71	532	3·41	3111	5·95	939	2·98	309	2·60	0·436
20/12	200	120	12	14	4·7	37·3	29·2	1546	6·45	428	3·39	3188	6·64	696	2·68	259	2·64	0·364
			14			43·2	33·9	1776	6·42	489	3·36	3725	6·72	818	2·76	295	2·61	0·362
			16			49·0	38·4	1997	6·38	547	3·34	4264	6·80	941	2·84	331	2·60	0·360
20/15	200	150	12	17	5·7	41·0	32·2	1667	6·37	810	4·44	3181	6·08	1347	3·62	436	3·26	0·552
			16			53·9	42·3	2155	6·32	1043	4·40	4272	6·27	1812	3·78	562	3·23	0·549
			18			60·3	47·3	2388	6·30	1153	4·38	4800	6·33	2048	3·85	623	3·21	0·548
			20			66·5	52·2	2614	6·27	1258	4·35	5344	6·41	2285	3·93	683	3·21	0·546

TABLE 3. I-BEAMS



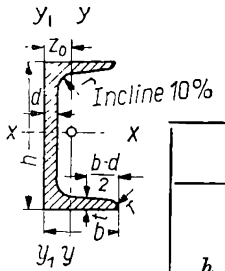
Sec. No.	Wt./m in kg	Dimensions in mm						Cross-sectional area F in cm	Axes							Sec. No.
									xx				yy			
		h	b	d	t	r	r_1		J_x, cm^4	W_x, cm^3	i_x, cm	$J_x : S_x$	J_y, cm^4	W_y, cm^3	i_y, cm	
10	11·2	100	68	4·5	7·6	6·5	3·3	14·3	245	49	4·14	8·59	33·0	9·72	1·52	10
12	14·0	120	74	5·0	8·4	7·0	3·5	17·8	436	72·7	4·95	10·3	46·9	12·7	1·62	12
14	16·9	140	80	5·5	9·1	7·5	3·8	21·5	712	102	5·76	12·0	64·4	16·1	1·73	14
16	20·5	160	88	6·0	9·9	8·0	4·0	26·1	1130	141	6·58	13·8	93·1	21·2	1·89	16
18	24·1	180	94	6·5	10·7	8·5	4·3	30·6	1660	185	7·36	15·4	122	26·0	2·00	18
20 a	27·9	200	100	7·0	11·4	9·0	4·5	35·5	2370	237	8·15	17·2	158	31·5	2·12	a 20
20 b	31·1	200	102	9·0	11·4	9·0	4·5	39·5	2500	250	7·96	16·9	169	33·1	2·06	b
22 a	33·0	220	110	7·5	12·3	9·5	4·8	42·0	3400	309	8·99	18·9	225	40·9	2·31	a 22
22 b	36·4	220	112	9·5	12·3	9·5	4·8	46·4	3570	325	8·78	18·7	239	42·7	2·27	b

TABLE 3—*contd.*

Sec. No.	Wt./m in kg	Dimensions in mm						Cross-sectional area F in cm²	Axes							Sec. No.
									xx				yy			
		h	b	d	t	r	r_1		J_x, cm⁴	W_x, cm³	i_x, cm	$J_x : S_x$	J_y, cm⁴	W_y, cm³	i_y, cm	
24 a	37·4	240	116	8·0	13·0	10·0	5·0	47·7	4570	381	9·77	20·7	280	48·4	2·42	a
b	41·2	240	118	10·0	13·0	10·0	5·0	52·6	4800	400	9·57	20·4	297	50·4	2·38	b 24
27 a	42·8	270	122	8·5	13·7	10·5	5·3	54·6	6550	485	10·9	23·8	345	56·6	2·51	a
b	47·1	270	124	10·5	13·7	10·5	5·3	60·0	6870	509	10·7	22·9	366	58·9	2·47	b 27
a	48·0	300	126	9·0	14·4	11·0	5·5	61·2	8950	597	12·1	25·7	400	63·5	2·55	a
30 b	52·7	300	128	11·0	14·4	11·0	5·5	67·2	9400	627	11·8	25·4	422	65·9	2·50	b 30
c	57·4	300	130	13·0	14·4	11·0	5·5	73·4	9850	657	11·6	25·0	445	68·5	2·46	c
a	53·4	330	130	9·5	15·0	11·5	5·8	68·1	11,900	721	13·2	28·3	460	70·7	2·60	a
33 b	58·6	330	132	11·5	15·0	11·5	5·8	74·7	12,500	757	12·9	27·9	484	73·4	2·55	b 33
c	63·8	330	134	13·5	15·0	11·5	5·8	81·3	13,100	794	12·7	27·5	510	76·1	2·51	c

	a	59·9	360	136	10·0	15·8	12·0	6·0	76·3	15,760	875	14·4	30·7	552	81·2	2·69	a
36	b	65·6	360	138	12·0	15·8	12·0	6·0	83·5	16,530	919	14·1	30·3	582	84·3	2·64	b 36
	c	71·2	360	140	14·0	15·8	12·0	6·0	90·7	17,310	962	13·8	29·9	612	87·4	2·60	c
	a	67·6	400	142	10·5	16·5	12·5	6·3	86·1	21,720	1090	15·9	39·1	660	93·2	2·77	a
40	b	73·8	400	144	12·5	16·5	12·5	6·3	94·1	22,780	1140	15·6	33·6	692	96·2	2·71	b 40
	c	80·1	400	146	14·5	16·5	12·5	6·3	102	23,850	1190	15·2	33·2	727	99·6	2·65	c
	a	80·4	450	150	11·5	18	13·5	6·8	102	32,240	1430	17·7	38·6	855	114	2·89	a
45	b	87·4	450	152	13·5	18	13·5	6·8	111	33,760	1500	17·4	38·0	894	118	2·84	b 45
	c	94·5	450	154	15·5	18	13·5	6·8	120	35,280	1570	17·1	37·6	938	122	2·79	c
	a	93·6	500	158	12	20	14	7·0	119	46,470	1860	19·7	42·8	1120	142	3·07	a
50	b	101	500	160	14	20	14	7·0	129	48,560	1940	19·4	42·4	1170	146	3·01	b 50
	c	109	500	162	16	20	14	7·0	139	50,640	2030	19·0	41·8	1220	151	2·96	c
	a	105	550	166	12·5	21	14·5	7·3	134	62,810	2290	21·6	46·9	1370	164	3.19	a
55	b	114	550	168	14·5	21	14·5	7·3	145	65,640	2390	21·2	46·4	1420	170	3·14	b 55
	c	123	550	170	16·5	21	14·5	7·3	156	68,410	2490	20·9	45·8	1480	175	3·08	c
	a	118	600	176	13	22	15	7·5	151	83,860	2800	23·5	51·8	1700	193	3·36	a
60	b	128	600	178	15	22	15	7·5	163	87,460	2920	23·2	50·7	1770	199	3·30	b 60
	c	137	600	180	17	22	15	7·5	175	91,060	3040	22·8	50·2	1840	205	3·24	c

TABLE 4. CHANNELS



Sec. No.	Wt./m in kg	Dimensions in mm						Cross-sectional area F, in cm²	Axes							
									xx			yy			y_1y_1	z_0,
		h	b	d	t	r	r_1		J_x, cm⁴	W_x, cm³	i_x, cm	J_y, cm⁴	W_y, cm³	i_y, cm	J_{y_1}, cm⁴	cm
5	5·44	50	37	4·5	7·0	7·0	3·5	6·93	26·0	10·4	1·94	8·3	3·55	1·10	20·9	1·35
6·5	6·70	65	40	4·8	7·5	7·5	3·75	8·54	55·2	17·0	2·54	12·0	4·59	1·19	28·3	1·38
8	8·04	80	43	5·0	8·0	8·0	4·00	10·24	101·3	25·3	3·15	16·6	5·79	1·27	37·4	1·43
10	10·00	100	48	5·3	8·5	8·5	4·25	12·74	198·3	39·7	3·95	25·6	7·80	1·41	54·9	1·52
12	12·06	120	53	5·5	9·0	9·0	4·50	15·36	346·3	57·7	4·75	37·4	10·17	1·56	77·7	1·62
14 a	14·53	140	58	6·0	9·5	9·5	4·75	18·51	563·7	80·5	5·52	53·2	13·01	1·70	107·1	1·71
14 b	16·73	140	60	8·0	9·5	9·5	4·75	21·31	609·4	87·1	5·35	61·1	14·12	1·69	120·6	1·67
16 a	17·23	160	63	6·5	10·0	10·0	5·0	21·95	866·2	108·3	6·28	73·3	16·30	1·83	144·1	1·80
16 b	19·74	160	65	8·5	10·0	10·0	5·0	25·15	934·5	116·8	6·10	83·4	17·55	1·82	160·8	1·75
18 a	20·17	180	68	7·0	10·5	10·5	5·25	25·69	1272·7	141·4	7·04	98·6	20·03	1·96	189·7	1·88
18 b	22·99	180	70	9·0	10·5	10·5	5·25	29·29	1369·9	152·2	6·84	111·0	21·52	1·95	210·1	1·84
20 a	22·63	200	73	7·0	11·0	11·0	5·5	28·83	1780·4	178·0	7·86	128·0	24·20	2·11	244·0	2·01
20 b	25·77	200	75	9·0	11·0	11·0	5·5	32·83	1913·7	191·4	7·64	143·6	25·88	2·09	268·4	1·95

22	a	24·99	220	77	7·0	11·5	11·5	5·75	31·84	2,393·9	217·6	8·67	157·8	28·17	2·23	298·2	2·10
	b	28·45	220	79	9·0	11·5	11·5	5·75	36·24	2,571·4	233·8	8·42	176·4	30·05	2·21	326·3	2·03
	a	26·55	240	78	7·0	12·0	12·0	6·0	34·21	3,052·2	254·3	9·45	173·8	30·47	2·25	324·6	2·10
24	b	30·62	240	80	9·0	12·0	12·0	6·0	39·00	3,282·6	273·5	9·17	194·1	32·51	2·23	354·8	2·03
	c	34·39	240	82	11·0	12·0	12·0	6·0	43·81	3,513·0	292·7	8·96	213·4	34·42	2·21	388·1	2·00
	a	30·83	270	82	7·5	12·5	12·5	6·25	39·27	4,362·0	323·1	10·54	215·6	35·52	2·34	393·1	2·13
27	b	35·07	270	84	9·5	12·5	12·5	6·25	44·67	4,690·1	347·4	10·25	239·2	37·72	2·31	428·2	2·06
	c	39·30	270	86	11·5	12·5	12·5	6·25	50·07	5,018·1	371·7	10·10	264·4	39·79	2·28	466·8	2·03
30	a	34·45	300	85	7·5	13·5	13·5	6·75	43·89	6,047·9	403·2	11·72	259·5	41·10	2·43	466·5	2·17
	b	39·16	300	87	9·5	13·5	13·5	6·75	49·59	6,497·9	433·2	11·41	289·2	44·03	2·41	515·2	2·13
	c	43·87	300	89	11·5	13·5	13·5	6·75	55·89	6,947·9	463·2	11·15	315·8	46·38	2·38	559·7	2·09
	a	38·70	330	88	8·0	14·0	14·0	7·0	49·50	8,076·8	489·5	12·80	307·5	46·65	2·50	547·6	2·21
33	b	43·88	330	90	10·0	14·0	14·0	7·0	55·90	8,675·7	525·8	12·46	338·4	49·32	2·46	593·2	2·14
	c	49·06	330	92	12·0	14·0	14·0	7·0	62·50	9,274·7	562·1	12·18	367·9	51·81	2·43	643·2	2·10
	a	47·80	360	96	9·0	16·0	16·0	8·0	60·89	11,874·2	659·7	13·97	455·0	63·54	2·73	818·4	2·44
36	b	53·45	360	98	11·0	16·0	16·0	8·0	68·09	12,651·8	702·9	13·63	496·7	66·85	2·70	880·4	2·37
	c	59·10	360	100	13·0	16·0	16·0	8·0	75·29	13,429·4	746·1	13·36	536·4	70·02	2·67	947·9	2·34
	a	58·91	400	100	10·5	18·0	18·0	9·0	75·05	17,577·9	878·9	15·30	592·0	78·83	2·81	1067·7	2·49
40	b	65·19	400	102	12·5	18·0	18·0	9·0	83·05	18,644·5	932·2	14·98	640·0	82·52	2·78	1135·6	2·44
	c	71·47	400	104	14·5	18·0	18·0	9·0	91·05	19,711·2	985·6	14·71	687·8	86·19	2·75	1220·7	2·42

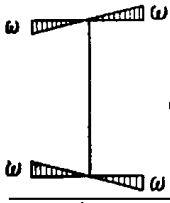
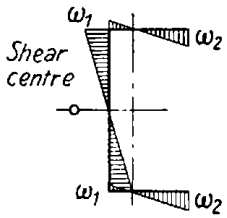


TABLE 5. SECTORIAL CHARACTERISTICS OF ROLLED I-SECTIONS

(OST 10,016-39)

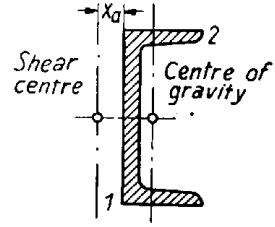
Sec. No.	Sectorial second moment of area J_ω, cm^6	Sectorial area for extreme point ω_{max}, cm^2	Sectorial section modulus W_ω, cm^4	Second moment of area for pure torsion J_k, cm^4	Elastic flexuro-torsional characteristic $\alpha = \sqrt{\frac{EJ_t}{GJ_\omega}}$, cm^{-1}
10	644·3	15·25	42·26	2·873	0·041 22
12	1,353	20·10	67·33	4·243	0·034 57
14	2,560	25·54	100·23	5·911	0·029 66
16	4,879	32·25	151·30	8·406	0·025 62
18	8,219	38·90	211·28	11·37	0·022 95
20 a	13,121	46·15	284·31	14·81	0·020 74
b	13,857	47·05	294·50	17·85	0·022 15
22 a	22,773	55·91	407·33	20·32	0·018 44
b	23,930	56·90	420·55	24·08	0·019 58
24 a	33,799	64·48	524·15	25·57	0·016 98
b	35,426	65·57	540·25	30·12	0·018 00
27 a	52,987	76·68	690·99	31·93	0·015 15
b	55,414	77·92	711·21	37·60	0·016 08
a	76,704	88·38	867·93	38·83	0·013 89
30 b	80,114	89·75	892·60	45·78	0·014 75
c	83,612	91·13	917·50	55·23	0·015 87
a	107,160	100·69	1064·3	46·19	0·012 81
33 b	111,780	102·21	1093·6	54·49	0·013 63
c	116,520	103·73	1123·3	65·74	0·014 66
a	154,820	115·19	1344·0	56·85	0·011 83
36 b	161,210	116·85	1379·6	66·72	0·012 56
c	167,760	118·51	1415·6	79·99	0·013 48
a	228,900	134·13	1706·6	68·75	0·010 70
40 b	237,950	136·00	1749·6	80·68	0·011 37
c	247,210	137·85	1793·3	96·55	0·012 20
a	376,630	159·75	2357·6	95·31	0·009 819
45 b	390,770	161·86	2414·4	111·3	0·010 41
c	405,220	163·96	2471·5	131·8	0·011 13
a	611,990	187·10	3270·9	131·2	0·009 038
50 b	633,900	189·44	3346·2	150·3	0·009 504
c	656,210	191·97	3421·8	174·9	0·010 07
a	906,350	216·79	4180·8	159·9	0·008 198
55 b	937,220	219·36	4272·5	182·7	0·008 617
c	968,720	221·94	4364·8	211·5	0·009 119
a	1,349,900	251·22	5373·4	195·5	0·007 427
60 b	1,393,200	254·04	5484·2	221·9	0·007 790
c	1,437,300	256·86	5595·7	255·3	0·008 226

Note: In calculating α take $G = 800{,}000$ kg/cm^2; $E = 2{,}100{,}000$ kg/cm^2.



APPENDIX A

TABLE 6.
SECTORIAL CHARACTERISTICS OF ROLLED CHANNELS
(OST 10,016-39)



Sec. No.	Co-ordinate x_a of shear centre, cm	Sectorial second moment of area J_ω, cm^6	Sectorial areas		Sectorial section moduli		Second moment of area J_t for pure torsion, cm^4	Elastic flexuro-torsional characteristic $\alpha = \sqrt{\frac{GJ_t}{EJ_\omega}}$, cm^{-1}
			ω_1, cm^2	ω_2, cm^2	$W_{\omega 1}$, cm^4	$W_{\omega 2}$, cm^4		
5	1·08	24·91	2·70	4·26	9·22	5·85	1·350	0·1437
6·5	1·15	64·88	3·86	6·36	16·80	10·21	1·497	0·09375
8	1·22	141·8	5·15	8·75	27·57	16·20	1·940	0·07219
10	1·34	354·8	7·19	12·71	49·35	27·92	2·727	0·05411
12	1·48	768·3	9·54	17·31	80·51	44·39	3·634	0·04245
14 a	1·58	1,512	12·03	22·63	125·74	66·85	4·815	0·03483
14 b	1·39	1,711	11·46	23·85	149·32	71·75	6·248	0·03730
16 a	1·68	2,760	14·74	28·63	187·23	96·40	6·306	0·02950
16 b	1·48	3,099	14·03	30·09	220·87	103·00	8·227	0·03180
18 a	1·83	4,745	17·68	35·32	268·41	134·34	8·128	0·02555
18 b	1·57	5,292	16·83	37·02	314·50	142·95	10·50	0·02749
20 a	1·94	7,698	21·27	42·46	361·95	181·28	9·84	0·02207
20 b	1·73	8,560	20·24	44·45	422·87	192·57	12·50	0·02359
22 a	2·07	11,593	24·84	49·60	466·69	233·73	11·66	0·01958
22 b	1·86	12,863	23·63	51·88	544·42	247·95	14·60	0·02079
24 a	2·10	15,326	27·48	55·21	557·74	277·59	13·21	0·01812
24 b	1·88	17,007	26·10	57·75	651·56	294·50	16·47	0·01921
24 c	1·67	18,640	24·91	60·09	748·35	310·21	21·31	0·02087
27 a	2·14	24,337	31·85	66·46	764·11	366·19	16·25	0·01595
27 b	1·91	26,883	30·23	69·39	889·34	387·42	20·34	0·01698
27 c	1·70	29,355	28·82	72·10	1018·6	407·12	26·34	0·01848
30 a	2·26	36,645	37·21	76·54	984·87	478·78	20·39	0·01456
30 b	2·03	40,436	35·23	79·98	1147·8	505·61	25·01	0·01535
30 c	1·80	44,104	33·59	83·06	1313·0	530·97	31·75	0·01656
33 a	2·25	52,630	41·39	88·54	1271·7	594·43	24·29	0·01326
33 b	2·02	57,844	39·27	92·27	1473·2	626·93	29·92	0·01404
33 c	1·80	62,890	37·44	95·69	1679·8	657·23	38·04	0·01518
36 a	2·47	92,189	49·50	104·55	1862·2	881·77	38·91	0·01268
36 b	2·24	100,430	47·30	108·51	2123·4	925·54	46·56	0·01329
36 c	2·02	108,420	45·36	112·18	2390·2	966·48	57·18	0·01417
40 a	2·43	148,100	55·78	121·67	2655·1	1217·2	59·74	0·01240
40 b	2·21	160,100	53·51	125·86	2991·7	1272·1	70·78	0·01298
40 c	2·00	171,870	51·51	129·80	3336·4	1324·0	85·72	0·01378

Note: In calculating α take $G = 800{,}000$ kg/cm^2; $E = 2{,}100{,}000$ kg/cm^2.

TABLE 7. Φ-COEFFICIENTS

Slenderness ratio $\lambda = \frac{\mu l}{i}$	Values of Φ for				
	Steels of mark 4, 3, 2 OS	Steel of mark 5	Steel SPB	Cast iron	Timber
0	1·00	1·00	1·00	1·00	1·00
10	0·99	0·98	0·97	0·97	0·99
20	0·96	0·95	0·95	0·91	0·97
30	0·94	0·92	0·91	0·81	0·93
40	0·92	0·89	0·87	0·69	0·87
50	0·89	0·86	0·83	0·57	0·80
60	0·86	0·82	0·79	0·44	0·71
70	0·81	0·76	0·72	0·34	0·60
80	0·75	0·70	0·65	0·26	0·48
90	0·69	0·62	0·55	0·20	0·38
100	0·60	0·51	0·43	0·16	0·31
110	0·52	0·43	0·35	—	0·25
120	0·45	0·36	0·30	—	0·22
130	0·40	0·33	0·26	—	0·18
140	0·36	0·29	0·23	—	0·16
150	0·32	0·26	0·21	—	0·14
160	0·29	0·24	0·19	—	0·12
170	0·26	0·21	0·17	—	0·11
180	0·23	0·19	0·15	—	0·10
190	0·21	0·17	0·14	—	0·09
200	0·19	0·16	0·13	—	0·08

APPENDIX B

1. CONVERSION TABLES

METRIC TO BRITISH

$1\ \text{m} = 39{\cdot}370\ \text{in.} = 3{\cdot}2808\ \text{ft}$
$1\ \text{m}^2 = 10{\cdot}764\ \text{ft}^2$
$1\ \text{m}^3 = 35{\cdot}315\ \text{ft}^3$

$1\ \text{cm} = 0{\cdot}39370\ \text{in.}$
$1\ \text{cm}^2 = 0{\cdot}15500\ \text{in}^2$
$1\ \text{cm}^3 = 0{\cdot}061024\ \text{in}^3$
$1\ \text{cm}^4 = 0{\cdot}024025\ \text{in}^4$
$1\ \text{cm}^5 = 0{\cdot}0094587\ \text{in}^5$
$1\ \text{cm}^6 = 0{\cdot}0037239\ \text{in}^6$
$1\ \text{cm}^{-1} = 2{\cdot}5400\ \text{in}^{-1}$

$1\ \text{t} = 2204{\cdot}6\ \text{lb}$
$1\ \text{kg} = 2{\cdot}2046\ \text{lb}$

$1\ \text{kg/m} = 0{\cdot}67197\ \text{lb/ft}$
$1\ \text{kg/cm} = 5{\cdot}5997\ \text{lb/in.}$

$1\ \text{kg/m}^2 = 0{\cdot}20482\ \text{lb/ft}^2$
$1\ \text{kg/cm}^2 = 14{\cdot}223\ \text{lb/in}^2$
$1\ \text{kg/mm}^2 = 1422{\cdot}3\ \text{lb/in}^2 = 0{\cdot}63496\ \text{ton/in}^2$

$1\ \text{t/m}^3 = 1\ \text{g/cm}^3 = 0{\cdot}036127\ \text{lb/in}^3 = 62{\cdot}427\ \text{lb/ft}^3$

$1\ \text{tm} = 3{\cdot}2290\ \text{tonft}$
$1\ \text{kgcm} = 0{\cdot}86796\ \text{lbin.}$

$1\ \text{kgcm}^2 = 0{\cdot}34172\ \text{lbin}^2$
$1\ \text{kgcm}^3 = 0{\cdot}13453\ \text{lbin}^3$
$1\ \text{cm/kg} = 0{\cdot}17858\ \text{in/lb}$

BRITISH TO METRIC

1 ft $= 30{\cdot}480$ cm $= 0{\cdot}304\,80$ m
1 ft^2 $= 929{\cdot}03$ cm^2 $= 0{\cdot}092\,903$ m^2
1 ft^3 $= 28{,}317$ cm^3 $= 0{\cdot}028\,317$ m^3

1 in. $= 25{\cdot}400$ mm $= 2{\cdot}5400$ cm
1 in^2 $= 645{\cdot}16$ mm^2 $= 6{\cdot}4516$ cm^2
1 in^3 $= 16{\cdot}387$ cm^3
1 in^4 $= 41{\cdot}623$ cm^4
1 in^5 $= 105{\cdot}72$ cm^5
1 in^6 $= 268{\cdot}53$ cm^6
1 in^{-1} $= 0{\cdot}393\,70$ cm^{-1}

1 ton $= 1016{\cdot}0$ kg $= 1{\cdot}0160$ t
1 lb $= 453{\cdot}59$ g $= 0{\cdot}453\,59$ kg

1 lb/ft $= 1{\cdot}488\,2$ kg/m
1 lb/in. $= 0{\cdot}178\,58$ kg/cm
1 lb/ft^2 $= 4{\cdot}882\,3$ kg/m^2
1 lb/in^2 $= 0{\cdot}070\,308$ kg/cm^2
1 lb/ft^3 $= 0{\cdot}016\,018$ g/cm^3
1 lb/in^3 $= 27{\cdot}680$ g/cm^3

1 ton ft $= 0{\cdot}309\,69$ t m
1 lb in. $= 1{\cdot}1521$ kg cm
1 lb in^2 $= 2{\cdot}9264$ kg cm^2
1 lb in^3 $= 7{\cdot}4330$ kg cm^3
1 in/lb $= 5{\cdot}5997$ cm/kg

2. THEORIES OF STRENGTH *†

Let σ_1, σ_2 and σ_3 ($\sigma_1 > \sigma_2 > \sigma_3$) be the principal stresses, and ε_1, ε_2 and ε_3 the principal strains of a material in the general, three-

* See Chapter 2 of text.

† Only the four most important (from a practical viewpoint) theories of strength are considered here. For brevity, they are referred to as the first, second, third, and fourth theories respectively. For a comprehensive treatment of this subject the reader is referred to *Advanced Mechanics of Materials* by G. Murphy (McGraw-Hill), or any other textbook on advanced strength of materials.

dimensional state of stress. Further, let σ and ε be the stress and the strain respectively of the same material subjected to an axial test. Then the four most important criteria of the limiting strength of the material may be expressed in the given terms as follows.

(1) The first theory of strength (or the theory of maximum normal stress) is stated as

$$\sigma_1 \leq [\sigma],$$

or

$$|\sigma_3| \leq [\sigma]$$

where $[\sigma]$ is the maximum allowable value of σ. This theory agrees with practical results only for brittle materials.

(2) The second theory of strength (or the theory of maximum normal strain) is stated as

$$\varepsilon_1 = \varepsilon_{\max} \leq [\varepsilon],$$

where $[\varepsilon]$ is the maximum allowable value of ε, i.e. $[\varepsilon] = [\sigma]/E$, where E is the modulus of elasticity of the material. Now, in a three-dimensional state of stress ε_1 is expressed as

$$\varepsilon_1 = \frac{1}{E}\{\sigma_1 - \mu(\sigma_2 + \sigma_3)\},$$

where μ is Poisson's ratio. Thus, equating the two expressions for ε_1, we have (after eliminating E)

$$\{\sigma_1 - \mu(\sigma_2 + \sigma_3)\} \leq [\sigma].$$

Thus, in applying the theory of maximum normal strain, it is necessary to compare the totality of the three principal stresses with the maximum allowable stress $[\sigma]$ established in an axial test. This totality of principal stresses is termed the effective stress, and expressed as

$$\sigma_{r2} = \{\sigma_1 - \mu(\sigma_2 + \sigma_3)\},$$

where suffix 2 indicates that this effective stress is based on the second theory of strength. This theory is in good agreement with practical results for brittle materials.

(3) The third theory of strength (or the maximum shear stress theory) is expressed as

$$\tau_{\max} \leq [\tau]$$

where $[\tau]$ is the maximum allowable shear stress as established in an axial test, and $\tau_{\max}$ the maximum value of the actual shear stress of the material in a three-dimensional state of stress. It can be shown that

$$[\tau] = \frac{[\sigma]}{2}$$

and

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}.$$

Thus, the effective stress, in accordance with the third theory of strength, is expressed as

$$\sigma_{r3} = \sigma_1 - \sigma_3.$$

This theory agrees satisfactorily with most practical results, especially in cases of ductile materials, although there are some notable exceptions.

(4) The fourth theory of strength (or the theory of maximum shear strain energy) is stated as

$$U_{sh} \leq [U_{sh}]$$

where U_{sh} and $[U_{sh}]$ are the shear strain energy of a material in general, three-dimensional state of stress, and the maximum shear strain energy of the same material in an axial test respectively. It can be shown (subtracting from the total strain energy the portion due to the change in the volume of the material) that

$$U_{sh} = \frac{1+\mu}{3E}\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3\},$$

and

$$[U_{sh}] = \frac{1+\mu}{3E}[\sigma]^2.$$

Thus, the state of limiting strength of the material may be expressed as

$$\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3\}^{\frac{1}{2}} \leq [\sigma],$$

or

$$\left\{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]\right\}^{\frac{1}{2}} \leq [\sigma].$$

The effective stress σ_{r4} is given as

$$\sigma_{r4} = \{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3\}^{\frac{1}{2}}.$$

The fourth theory of strength is the most accurate theory for ductile materials. Further, it can be shown that this theory is valid for the condition of hydrostatic pressure, which is not true for the other theories.

3. VERESHCHAGIN'S METHOD*

This method, which was suggested by A. N. Vereshchagin in 1924, is a development of the method of virtual work. The method of virtual work (described by Clerk Maxwell in 1864, and by Otto Mohr in 1874, and therefore sometimes also known as the Maxwell–Mohr method) expresses deflection y of a structure in the form of the following integral

$$y = \int_0^l \frac{M(x)[\partial M(x)/\partial P]}{EJ}\,\mathrm{d}x,$$

where $M(x)$ is the bending moment at distance x from the origin, and the quantity $\partial M(x)/\partial P$ is the rate of change of this bending moment due to a change in the load P at the point where the deflection is desired. (EJ designates the flexural rigidity of the structure.) It can be seen that $\partial M(x)/\partial P$ is equal to the bending moment due to a unit load at the point where the deflection is

* See Chapter 5 of text.

desired. Designating this bending moment by M^0, the expression for the deflection is written as

$$y = \int_0^l \frac{M(x)\,M^0}{EJ}\,\mathrm{d}x.$$

Vereshchagin's method in essence consists of a simplification in evaluating the above integral. It is most easily understood by considering the figure given below.

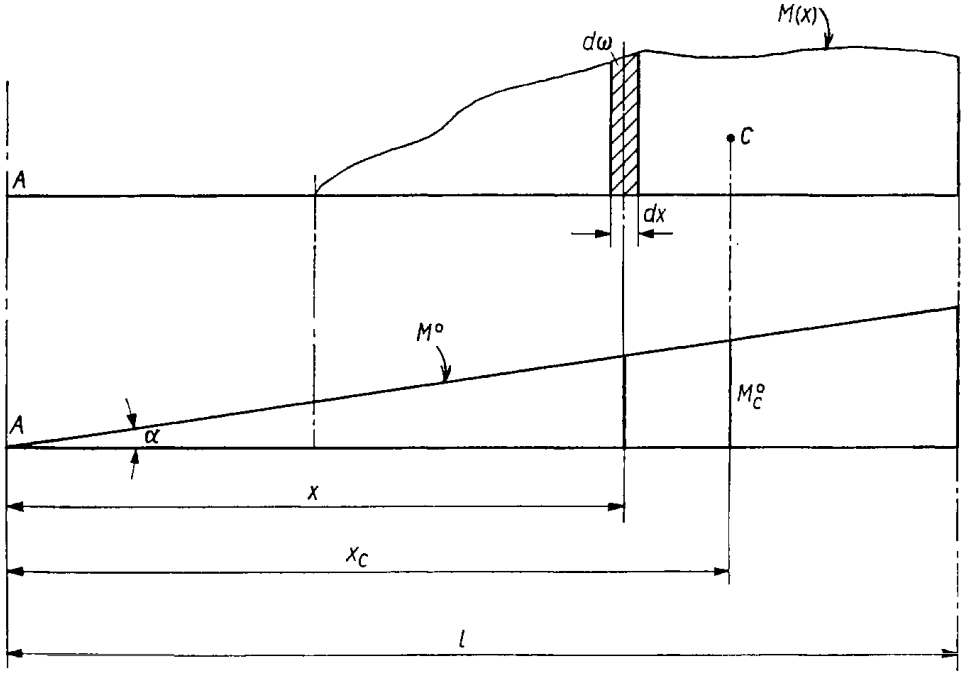


FIG. 1

The contour of $M(x)$ can have any shape, whereas that of M^0 is given by a straight line. The product $M(x)\,\mathrm{d}x$ may be considered as an element $\mathrm{d}\omega$ of the area of the bending moment $M(x)$ diagram taken over a distance $\mathrm{d}x$ (shown hatched in the figure).

The ordinate of the bending moment M^0 diagram at distance x from point A (the deflection of which is being determined) is given by $M^0 = x \tan \alpha$. Therefore the product $M(x)\,M^0\mathrm{d}x = \mathrm{d}\omega\, x \tan \alpha$,

and the whole integral $\int_0^l M(x) M^0 \mathrm{d}x = \tan\alpha \int_0^l \mathrm{d}\omega x$ represents the first moment of the bending moment $M(x)$ diagram area about point A, multiplied by $\tan\alpha$. But the first moment of the bending moment $M(x)$ diagram area is equal to the area of this diagram multiplied by the distance x_C of its centroid from point A. Thus,

$$\int_0^l M(x) M^0 \, \mathrm{d}x = \omega x_C \tan\alpha.$$

Now, it is easily seen that the quantity $x_C \tan\alpha$ is equal to the ordinate M_C^0 of the bending moment M^0 diagram, the position of which coincides with that of the centroid C of the $M(x)$ diagram. Hence the integral $\int_0^l M(x) M^0 \mathrm{d}x = \omega M_C$, and the deflection at point A is given as

$$y_A = \frac{\omega M_C^0}{EJ}.$$

Thus, in words: to determine the deflection y_A it is necessary to calculate the area ω of the bending moment $M(x)$ diagram, multiply it by the ordinate M_C^0 (the position of which, with respect to the x direction, coincides with the centroid of the $M(x)$ diagram) of the bending moment M^0 due to a unit load at A, and divide by the flexural rigidity EJ of the structure.

In order to determine the slope at point A, a unit couple instead of a unit load is assumed to be applied at this point. Employing the same reasoning as before, it can be shown that

$$\theta_A = \frac{\omega M_C^1}{EJ},$$

where θ_A is the slope at A, and M_C^1 the ordinate of the bending moment M^1 due to the unit couple at point A.

The method can be extended to cases where it is necessary to investigate the influence of axial forces $N(x)$, shear forces $Q(x)$ or torques M_t. In these cases the quantity EJ in the formula is replaced by EF, GF and GJ respectively, where F is the cross-sectional area of the member, G is the modulus of rigidity of the material, and J is the polar second moment of area of the member.

4. TORSION OF THIN-WALLED OPEN SECTIONS *

For a comprehensive treatment of this subject see *Thin Walled Elastic Beams* by Vlasov. A translation of this work was published by the Israel Program for Scientific Translations in 1961.

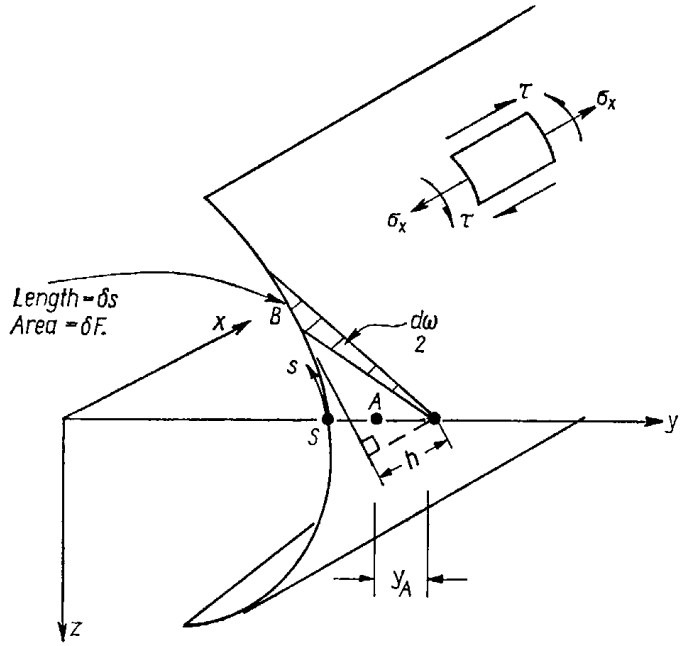


FIG. 2

Thin walled rods are defined as those where thicknesses are small compared with lateral dimensions and length. These notes refer to the examples in Section 30. All sections in the text are symmetrical about either the y or z axis. The notation is shown in the diagram.

The Sectorial Area (ω_0) at B with reference to any pole O, is defined as

$$\omega_0 = \sum_0^s h\delta s.$$

This is twice the area (shaded) swept out by radius vector OB in rotating form $s = 0$ to $s = s$.

* See Chapter 6 of text.

Shear Centre (A). This is the point about which the section rotates if the rod is subjected to pure torsion, there being no movement of that point in directions y or z.

If, as in the diagram, there is symmetry about the y axis, the position of A, which will lie on the axis of symmetry is given by

$$y_A = \frac{S_{\omega_0 z}}{J_y},$$

where J_y is the second moment of area of the cross-section about axis y, and

Sectorial-linear First Moment of Area ($S_{\omega_0 z}$) of the cross-section with respect to an arbitrarily chosen pole, O, on the y axis, and the origin for s is at S, on the y axis. It is given by

$$S_{\omega_0 z} = \sum^{F} \omega_0 z \delta F.$$

The Principal Sectorial Area (ω) at a point such as B is defined as the sectorial area having the pole at the shear centre A. The principal sectorial zero point is taken to lie on the y axis at S.

The Sectorial Second Moment of Area (J_ω). This is defined as follows

$$J_\omega = \sum^{F} \omega^2 \delta F.$$

Bimoment (B). Due to a pure torque in such a section there are longitudinal normal stresses (σ_z) set up. The bimoment, B, is defined by

$$B = \sum^{F} \sigma_x \omega \delta F,$$

or

$$\sigma_x = \frac{B_\omega}{I_\omega}.$$

(This may be compared with $\sigma_x = M_z/I_y$ for pure bending.) Similarly the shear stress (τ) is given by

$$\tau = -\frac{\dfrac{\mathrm{d}B}{\mathrm{d}x} S_\omega}{t I_\omega},$$

where $S_\omega = \sum_{0}^{S} \omega \mathrm{d}F.$

[This may be compared with shear stress distribution in beams subjected to shear force $\mathrm{d}M/\mathrm{d}x$, viz.

$$\tau = \left(\frac{\mathrm{d}M}{\mathrm{d}x}\right) \times \frac{A_y}{\mathrm{t}I_y}.]$$

The sum of the moments of the shear forces corresponding to these shear stresses about O is $\mathrm{d}B/\mathrm{d}x$ $(= M_\omega)$.

This is a "flexuro-twisting" moment—a twisting moment resisted by bending stresses.

The total torsional moment (T) on the section will consist of the unrestrained (St. Venant) torsion, H, and the flexuro-twisting moment (M_ω) where

$$H = GI_\mathrm{d} \frac{\mathrm{d}\theta}{\mathrm{d}x}.$$

θ is the angle of twist at any section.

For any set of loads and boundary conditions, T will be made up in different proportions of H and M_ω. It may be shown that

$$M_\omega = \frac{\mathrm{d}B}{\mathrm{d}x} = -EI_\omega \frac{\mathrm{d}^3\theta}{\mathrm{d}x^3}.$$

Thus

$$T = GI_\mathrm{d} \frac{\mathrm{d}\theta}{\mathrm{d}x} - EI_\omega \frac{\mathrm{d}^3\theta}{\mathrm{d}x^3}.$$

or

$$\frac{\mathrm{d}^4\theta}{\mathrm{d}x^4} - \frac{GI_\mathrm{d}}{EI_\omega} \times \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \frac{\mathrm{d}T}{\mathrm{d}x} \times \frac{1}{EI_\omega} = 0 \qquad (1)$$

or

$$\frac{\mathrm{d}^2B}{\mathrm{d}x^2} - \frac{GI_\mathrm{d}}{EI_\omega} B - \frac{\mathrm{d}T}{\mathrm{d}x} = 0.$$

Let

$$\alpha^2 = \frac{GI_\mathrm{d}}{EI_\omega}.$$

Equation (1) has as its solution,

$$\theta = A + Bx + C \cosh \alpha x + D \sinh \alpha x;$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = B + \alpha(C \sinh \alpha x + D \cosh \alpha x)$$

thus

$$B = EI_\omega \frac{d^2\theta}{dx^2} = -GI_d(C \cosh \alpha x + D \sinh \alpha x)$$

and

$$T = -EI_\omega \frac{d^3\theta}{dx^3} + GI_d\theta = GI_dB.$$

A, B, C, and D are arbitrary constants which may be determined for any particular loading case by substituting the relevant boundary conditions.

5. MECHANICAL PROPERTIES OF CERTAIN STEELS AND CAST IRONS MENTIONED IN PROBLEMS

Steel OS (St.—OS)
Yield stress = 19 kg/mm²
Ultimate strength = 32–47 kg/mm²

Steel 2 (St. 2)
Yield stress = 24 kg/mm²
Ultimate strength = 34–42 kg/mm²

Steel 3 (St. 3)
Yield stress = 22 kg/mm²
Ultimate strength = 38–47 kg/mm²

Steel 4 (St. 4)
Yield stress = 24 kg/mm²
Ultimate strength = 42–52 kg/mm²

Steel 5 (St. 5)
Yield stress = 27 kg/mm²
Ultimate strength = 50–62 kg/mm²

Ordinary grey cast iron
Ultimate strength in tension = 14–18kg/mm²
Ultimate strength in compression = 60–100 kg/mm²

Fine-grained grey cast iron
Ultimate strength in tension = 21–25 kg/mm²
Ultimate strength in compression = up to 140 kg/mm²

White cast iron
Ultimate strength in compression = up to 175 kg/mm²

6. NOTES ON TABLES AND DIAGRAMS

Taken from the book *Strength of Materials* by N. M. Belyayev
(12th Edition, 1959)

(1) Table 11 (pages 154, 155) deals with contact stresses, and is referred to in Problems 2·56 and 2.57.

(2) Table 17 (pages 214, 215) is concerned with torsion of non-circular profiles. It is referred to in Section 12.

(3) Table 30 (page 629) deals with critical loads on columns; it is referred to in Section 31.

(4) Table 33 (page 649) gives characteristic quantities for cross-sections, used in combined stability calculations. This table is referred to in Section 32.

(5) Table 37 (page 743) gives a number of stress concentration factors, necessary for solving problems in fatigue loading. It is referred to in Section 39.

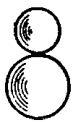
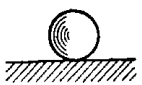
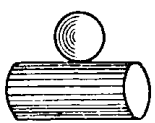
(6) Figure 626 (page 741) consists of notch sensitivity curves, necessary when dealing with fatigue loading. It is referred to in Section 39.

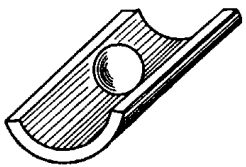
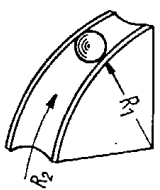
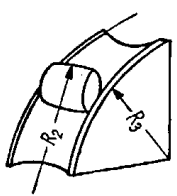
(7) Figure 629 (page 747) consists of scale factor curves; these are required for the solution of problems in fatigue loading. The table is referred to in Section 39.

(8) Figure 671 (page 816) deals with beams subjected to creep. It is referred to in Section 40.

Some of the diagrams have been supplemented with brief explanatory notes. These notes summarize in effect the essential references of the original text to these diagrams.

TABLE 11. FORMULAE FOR COEFFICIENTS A, B, AND σ_{max}

Shape of contacting bodies		Coefficients of the equation of the contact ellipse		Maximum stresses
Diagram of contact	Dimensions of contacting bodies	A	B	σ_{max}
	Two spheres of radii R_1 and R_2	$\frac{R_1 + R_2}{2R_1R_2}$	$\frac{R_1 + R_2}{2R_1R_2}$	$0{\cdot}388\sqrt[3]{PE^2\frac{(R_1 + R_2)^2}{(R_1R_2)^2}}$
	A sphere of radius R_1 and a hollow spherical surface of radius R_2	$\frac{R_2 - R_1}{2R_1R_2}$	$\frac{R_2 - R_1}{2R_2R_1}$	$0{\cdot}388\sqrt[3]{PE^2\frac{(R_2 - R_1)^2}{(R_1R_2)^2}}$
	A sphere of radius R and a plane ($R_2 = \infty$)	$\frac{1}{2R}$	$\frac{1}{2R}$	$0{\cdot}388\sqrt{PE^2\frac{1}{R^2}}$
	A sphere of radius R_1 and a cylinder of radius R_2 ($R_2 > R_1$)	$\frac{1}{2R_1}$	$\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$	$\alpha \times \sqrt[3]{\frac{PE^2}{R_1^2}}$

Shape of contacting bodies		Coefficients of the equation of the contact ellipse		Maximum stresses
Diagram of contact	Dimensions of contacting bodies	A	B	σ_{max}
	A sphere of radius R_1 and a cylindrical groove of radius R_2 ($R_2 > R_1$)	$\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$	$\frac{1}{2R_1}$	$\alpha \times \sqrt[3]{\frac{PE^2(R_2-R_1)^2}{(R_1R_2)^2}}$
	A sphere of radius R_1 and a circular race of radii R_2 and R_3 ($R_2 > R_3$)	$\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$	$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_3}\right)$	$\alpha \times \sqrt[3]{\frac{PE^2(R_2-R_1)^2}{(R_1R_2)^2}}$
	A bearing roller of radii R_1 and R_2, and a circular race of radii R_3 and R_4	$\frac{1}{2}\left(\frac{1}{R_2}-\frac{1}{R_4}\right)$	$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_3}\right)$	$\alpha \times \sqrt[3]{\frac{PE^2(R_4-R_2)^2}{(R_2R_4)^2}}$

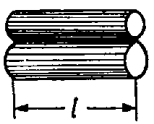
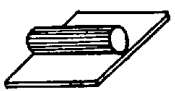
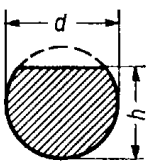
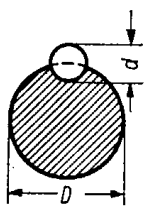
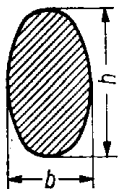
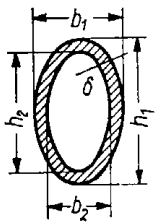
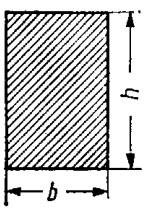
Shape of contacting bodies		Coefficients of the equation of the contact ellipse		Maximum stresses
Diagram of contact	Dimensions of contacting bodies	A	B	σ_{max}
	Two cylinders laid crosswise, having radii R_1 and R_2 ($R_2 > R_1$)	$\frac{1}{2R_2}$	$\frac{1}{2R_1}$	$\alpha \times \sqrt{\frac{PE^2}{R_2^2}}$
	Two cylinders with their axes parallel, having radii R_1 and R_2	—	$\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$	$0{\cdot}418\sqrt{\frac{PE}{l}\times\frac{R_1+R_2}{R_1R_2}}$
	A cylinder of radius R and a plane ($R_2 = \infty$)	—	$\frac{1}{2R}$	$0{\cdot}418\sqrt{\frac{PE}{l\times R}}$

TABLE 17. DATA ON TORSION OF NON-CIRCULAR PROFILES

Shape of cross-section	Torsional second moment of area of cross-section J_K — cm⁴	Torsional section modulus W_K — cm³	Points with maximum shear stresses $\max \tau = \frac{M_K}{W_K}$	Remarks
	$J_K = \frac{d^4}{16}\left(2{\cdot}6\,\frac{h}{d} - 1\right)$	$W_K = \frac{d^3}{8} \times \frac{\left(2{\cdot}6\,\frac{h}{d} - 1\right)}{0{\cdot}3\left(\frac{h}{d} + 0{\cdot}7\right)}$	$\max \tau = \frac{M_K}{W_K}$	$\frac{h}{d} > 0{\cdot}5$
	$J_K = \alpha \frac{D^4}{16}$	$W_K = \beta \frac{D^3}{8}$	At the bottom of the groove $\max \tau = \frac{M_K}{W_K}$	Values of coefficients α and β

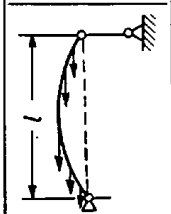
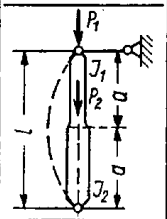
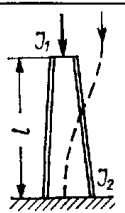
α and β from the table, depending on $\frac{d}{D}$

d/D	α	β	d/D	α	β
0·0	1·57	1·57	0·40	0·76	1·22
0·05	0·80	1·56	0·60	0·66	0·92
0·10	0·81	1·56	0·80	0·52	0·63
0·20	0·82	1·46	1·00	0·38	0·38

	$J_K = \frac{\pi}{16} \cdot \frac{m^3}{(m^2+1)} b^4$ $= \frac{16 F^4}{\pi^3 b h (b^2 + h^3)}$	$W_K = \frac{\pi b^3}{16} \times m$ $= \frac{\pi b^2 h}{16}$	(a) At the ends of the short axis $\tau_{max} = \frac{M_K}{W_K}$. (b) At the ends of the long axis $\tau = \frac{\tau_{Kmax}}{m}$	$\frac{h}{b} = m > 1$ F = cross-sectional area
	$J_K = \frac{\pi}{16} \frac{m^2}{m^2+1} b_1^4 (1-\alpha^4)$	$W_K = \frac{\pi b_1^3}{16} (1-\alpha^4)\, m$	(a) At the ends of short axis $\tau_{max} = \frac{M_K}{W_K}$. (b) At the ends of long axis $\tau = \frac{\tau_{max}}{m}$	$\frac{h_1}{b_1} = \frac{h_2}{b_2} = m > 1$, $\frac{h_2}{h_1} = \frac{b_2}{b_1} = \alpha < 1$. With a small thickness δ it is possible to assume a uniform distribution of the stresses along the cross-section $\tau = \frac{M_K}{2 F^0}$
	$J_K = a b^4$	$W_K = \beta b^3$	(a) At the mid-points of longer sides $\max \tau = \frac{M_K}{W_K}$. (b) At the mid-points shorter sides $\tau = \gamma \tau_{max}$ (c) In the corners the stresses are zero	Values of coefficients α, β and γ

$m = \frac{h}{b}$	α	β	γ
1·0	0·140	0·208	1·0
1·5	0·294	0·346	0·859
2·0	0·457	0·493	0·795
3·0	0·790	0·801	0·753
4·0	1·123	1·150	0·745

Table 30. Critical Forces

Type of loading	Diagram of column	Critical load	Length factors μ
Uniformly distributed along the length of the column (self-weight q)		$(ql)_k = \frac{\pi^3 EJ}{(\mu l)^2} = 0{\cdot}79\,\frac{\pi^2 EJ}{l^2}$	$\mu = 1{\cdot}12$
Uniformy distributed along the length of the column (self-weight q)		$(ql)_k = \frac{\pi^2 EJ}{(\mu l)^2} = 1{\cdot}9\,\frac{\pi^2 EJ}{l^2}$	$\mu = 0{\cdot}725$
Two forces P_1 and P_2 $\frac{P_1 + P_2}{P_1} = m$ $J_2 : J_1 = n$		$(P_1 + P_2)_k = \frac{\pi^2 EJ}{(\mu l)^2}$ μ taken from the table	See table below
(a) Concentrated force at the free end $J_1 : J_2 = n$ (b) Uniform cross-section (F = const) (c) (4 angles or like arrangements)		$P_k = \frac{\pi^2 FJ}{(\mu l)^2}$ μ taken from the table	See table below

n \ m	1·00	1·25	1·50	1·75	2·00	3·00
1·00	1·0	0·95	0·91	0·89	0·87	0·82
1·50	1·12	1·00	1·02	0·99	0·96	—
2·00	1·24	1·16	1·12	1·08	1·05	—

n	0·0	0·1	0·2	0·4	0·8	1·0
μ	6·25	2·71	2·42	2·28	2·07	2·0

TABLE 33. VALUES OF β IN THE FORMULA $P_{crit} = \beta/l^2 \sqrt{C_1 C_2}$

In this formula C_1 is the flexural rigidity, and C_2 the torsional rigidity of the beam. The coefficient β depends on the type of loading, the conditions of fixing, and the relationship $\alpha = C_1/C_2 \times l^2/h^2$, in which l is the span of the beam, and h the depth of its cross-section.

$\alpha = \frac{C_2}{C_1}\left(\frac{l}{h}\right)^2$	β_1	β_2	β_3	$\alpha = \frac{C_2}{C_1}\left(\frac{l}{h}\right)^2$	β_1	β_2	β_3
1	2	3	4	1	2	3	4
0·1	31·6	86·4	143·0	16	5·08	18·3	30·5
1·0	9·76	31·9	53·0	20	—	18·1	30·1
2·0	8·03	25·6	42·6	32	—	17·9	29·4
4·0	6·73	21·8	36·3	50	—	17·5	29·0
6·0	6·19	20·3	33·8	70	—	17·4	28·8
8·0	5·87	19·6	32·6	90	4·04	17·2	28·6
12·0	5·36	18·8	31·5	100	4·04	17·2	28·6

Column 1 refers to a cantilever loaded by a concentrated load at the free end.

Column 2 refers to a simply supported beam with a concentrated load at the mid-span.

Column 3 refers to a simply supported beam loaded by a uniformly distributed load.

TABLE 37. THEORETICAL STRESS CONCENTRATION FACTORS

Type of deformation, and causes for stress concentrations	α_K
I. Flexure and tension:	
1. Semi-circular recess on shaft, the ratio of the radius of the recess to the diameter of the shaft	
0·1	2·0
0·5	1·6
1·0	1·2
2·0	1·1
2. Fillet, with the ratio of the fillet radius to the depth of the cross-section (the diameter of the shaft)	
0·0625	1·75
0·125	1·50
0·25	1·20
0·5	1·10
3. Transition at a right angle	2·0
4. Sharp V-shaped recess	3·0
5. Screw thread (inch size)	2·0
6. Screw thread (metric)	2·5
7. Holes, with the ratio of the hole diameter to the transverse dimension of the section varying from 0·1 to 0·33	2·0
8. Machining marks on the surface of a component	1·2—1·4
II. Torsion:	
1. Fillet, with the ratio of the fillet radius to the least diameter of the shaft	
0·02	1·8
0·10	1·2
0·20	1·1
2. Keyways	1·6—2·0

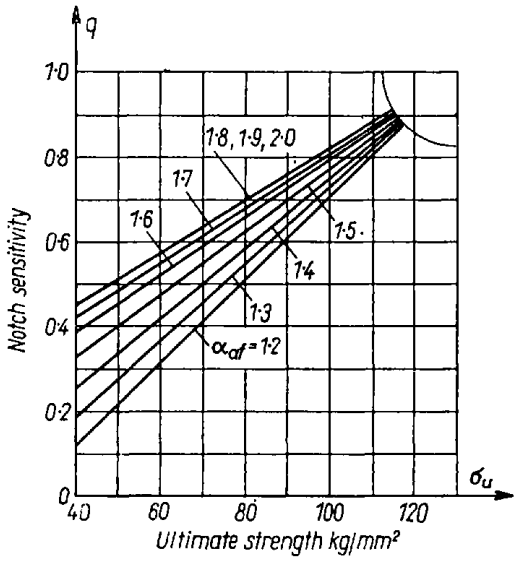


FIG. 626. Theoretical and experimental stress concentration factors. α_{tf} is the theoretical stress concentration factor, depending only on the shape of a component.

α_{af} is the actual stress concentration factor established by experiments, and depending on the shape and the material of a component.

The relationship between the notch sensitivity q, the theoretical stress concentration factor α_{tf}, and the actual stress concentration factor α_{af} is expressed as follows:

$$q = \frac{\alpha_{af} - 1}{\alpha_{tf} - 1}.$$

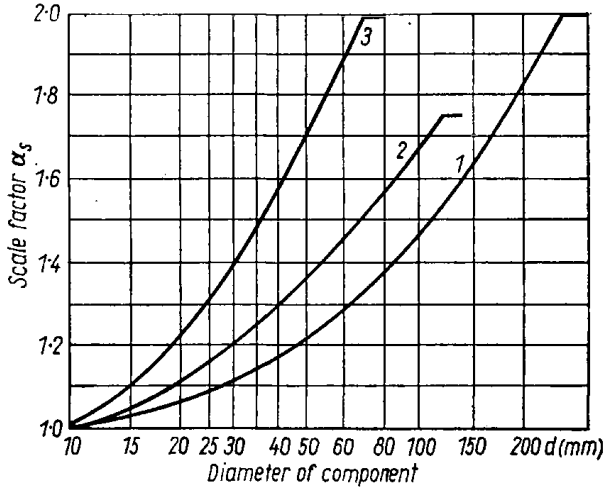


FIG. 629. Fatigue data

Curve 1 refers to components made of carbon steel, in the absence of stress concentrations.

Curve 2 refers to components made of carbon steel, in the presence of moderate stress concentrations (the stress concentration factor is less than 2).

Curve 3 refers to components made of alloy steel, in the absence of stress concentrations.

The relationship between the limit of endurance p_r^c of the component, the scale factor α_s, and the limit of endurance p_r of a small scale test-piece of the same material is expressed as follows:

$$p_r^c = \frac{p_r}{\alpha_s}.$$

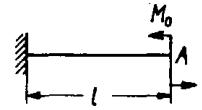
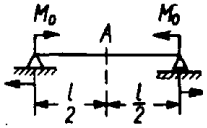
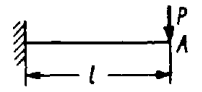
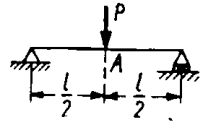
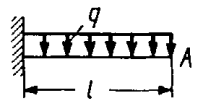
Diagrams of beams	$\frac{dy}{dt}\max = \left(\frac{dy}{dt}\right)_A = \frac{K}{J_{zn}^n}\Psi$ $\Psi =$
	$\frac{M_0^n l^3}{2}$
	$\frac{M_0^n l^2}{8}$
	$\frac{P^n l^{n+2}}{n+2}$
	$\frac{P^n l^{n+2}}{2^{2(n+1)}(n+2)}$
	$\frac{q^n l^{2(n+1)}}{(n+1)2^{n+1}}$

FIG. 671. Loading diagrams of beams in creep, and the corresponding values of the quantity Ψ. The rate of the maximum deflection in creep is given by the formula

$$\left(\frac{dy}{dt}\right)_{max} = \left(\frac{dy}{dt}\right)_A = \frac{k}{J_{zc}^n}\Psi.$$

In this formula k, n, stress σ and the steady creep rate v_c are connected by the relationship

$$v_c = k\sigma^n.$$

The quantity J_{zc}^n in creep is analogous to the second moment of area J_z of the cross-section of the beam in an elastic deformation.